WAVE DECAY WITH SINGULAR DAMPING

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Abstract. We consider the stabilization problem on a manifold with boundary for a wave equation with measure-valued linear damping. For a wide class of measures, containing Dirac masses on hypersurfaces as well as measures with fractal support, we establish an abstract energy decay result.

1. Introduction

In this paper, we consider a damped wave equation with measure-valued damping on a compact Riemannian manifold (Ω, g) with non-empty smooth boundary. If the measure is the Dirac mass on a hypersurface, then this problem has been considered on an interval in [BRT82, JTZ98] and on bounded domains in $\mathbb{R}^2$, provided the hypersurface stays away from the boundary. In particular, in [JTZ98], it is shown that generically for curved hypersurfaces, the energy of corresponding solutions decays to 0. However, if the domain is strictly convex, the existence of whispering gallery modes highly concentrated along the boundary [Ral69] shows that the decay rate is not uniform.

In this paper, we develop a functional analysis approach, assuming that the measure has good mapping properties between (generalized) Sobolev spaces. The novelty here is that the class of measures considered here is much wider than restrictions of the Riemannian volume to hypersurfaces (see Section 2). This mapping assumption allows us to prove an abstract energy decay result rather directly, in any dimension $n \geqslant 2$.

Let $-\Delta_g$ be the (positive) Laplace-Beltrami operator on Ω. Let dV_g denote the Riemannian volume element, and let $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ denote the L^2 inner product with respect to dV_g, with the convention that it is linear in the first argument. Since the boundary of Ω is non-empty, we prescribe Dirichlet boundary conditions to $-\Delta_g$, so that the domain is

$$D(-\Delta_g) = H^2(\Omega) \cap H_0^1(\Omega).$$

Here $H_0^1(\Omega)$ is the completion of $\mathcal{C}_c^\infty(\Omega)$ with respect to the homogeneous norm

$$\|u\|_{H_0^1(\Omega)} = \langle \nabla u, \nabla u \rangle_{L^2(\Omega)}^{1/2}.$$

We observe that the Dirichlet boundary conditions imply that for $u \in \mathcal{C}_c^\infty(\Omega)$,

$$0 = \|u\|_{H_0^1(\Omega)}^2 = \langle \nabla u, \nabla u \rangle_{L^2(\Omega)} \Leftrightarrow \ u \text{ is constant},$$

so this is a norm. In fact, we have

$$\|u\|_{H_0^1(\Omega)}^2 = \langle -\Delta u, u \rangle_{L^2(\Omega)} \geqslant \lambda_0 \|u\|_{L^2(\Omega)}^2,$$

where $\lambda_0 > 0$ is the first Dirichlet eigenvalue (necessarily positive).

Let $M : \mathcal{C}_c \to \mathcal{C}_c^*$ be a symmetric, non-negative linear operator which extends to a symmetric, non-negative, bounded mapping $M : H_0^1(\Omega) \to H^{-1}(\Omega)$. Such operators will be the most general "friction" coefficients we consider in this paper. For the functional analysis approach developed here, M need not be a local operator, but for applications we are most interested in the case where M is multiplication by a non-negative measure.

Let $\langle \cdot, \cdot \rangle_{H^{-1} \times H^1}$ be the duality product between $H^{-1}(\Omega)$ and $H^1(\Omega)$ so that for $g \in H^1(\Omega)$,

$$M(f)(g) = \langle M(f), g \rangle_{H^{-1} \times H^1}.$$

In order to use M in a wave equation with singular friction, we will write $\langle M(f), g \rangle$ as a shorthand for $\langle M(f), g \rangle = \langle M(f), \overline{g} \rangle_{H^{-1} \times H^1}$. In the case where μ is a non-negative measure on Ω, let $M = M_\mu$ denote the "multiplication" operator $M_\mu : \mathcal{C}_c \to \mathcal{C}_c^*$ given by

$$M_\mu(f)(g) = \int_\Omega f g d\mu, \qquad \langle M_\mu f, g \rangle = \int_\Omega f \bar{g} d\mu.$$

Through this paper, we will make the following central assumption :

$$M : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega) \text{ is a compact operator.} \tag{1.1}$$

Notice no assumption on the support of μ is made yet, or on the regularity. For example, μ need not be absolutely continuous with respect to dV_g : see Section (2) where several examples of such measures are given.

We will be interested in a wave equation with M-damping:

$$\begin{cases} (\partial_t^2 - \Delta + M\partial_t)u = 0, \text{ on } (0,\infty)_t \times \Omega \\ u|_{\partial\Omega} = 0, \text{ on } [0,\infty)_t \times \partial\Omega \\ u(0,x) = g(x), \ u_t(0,x) = f(x). \end{cases} \tag{1.2}$$

The natural space for the initial data (f, g) is*

$$\mathcal{H} = L^2(\Omega) \oplus H_0^1(\Omega).$$

We will say that $u \in C((0,\infty)_t)H_0^1(\Omega)$ is a weak solution to (1.2) if for every $\varphi \in H_0^1(\Omega)$,

$$\left\langle \partial_t^2 u, \varphi \right\rangle_{L^2(\Omega)} + \langle \nabla u, \nabla \varphi \rangle_{L^2(\Omega)} + \langle M\partial_t u, \varphi \rangle = 0.$$

As a first result, we prove the following:

Theorem 1. *Suppose the linear operator M extends to a bounded linear mapping $M : H_0^1(\Omega) \to H^{-1}(\Omega)$, and that this mapping is compact. Then G is maximally dissipative, and $(I - G)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. In particular, G generates a semigroup of contractions $e^{tG} : \mathcal{H} \to \mathcal{H}$, $t \geqslant 0$.*

*In fact, we will take a slightly smaller subspace for our initial data, namely $(f, g)^T \in D(G)$, where G is the matrix operator defined in (3.1).

Remark 2. As the proof will indicate, we will also describe the domain $D(G)$ by describing the domain of $(I-G)$. The strategy is to show that the real part of G is non-positive definite, then solve the Dirichlet problem. This will compute $D(G)$, show that $(I-G)$ is invertible on appropriate Hilbert spaces, and that the inverse is a compact operator.

The energy for this equation is the usual wave energy:

$$E(u,t) = \int_\Omega (|u_t|^2 + |\nabla u|^2) dV_g.$$

Since M is symmetric and non-negative, if $(f,g)^T \in D(G)$, then we compute

$$\begin{aligned} E'(u,t) &= 2\,\mathrm{Re} \int_\Omega \bar{u_t}(u_{tt} - \Delta_g u) dV_g \\ &= 2\,\mathrm{Re}\, \langle u_{tt} - \Delta_g u, u_t \rangle \\ &= 2\,\mathrm{Re}\, \langle -M u_t, u_t \rangle \\ &= -2 \langle M u_t, u_t \rangle \\ &\leqslant 0. \end{aligned}$$

It is a standard question to determine whether and how this energy decays to 0 asymptotically in time. As usual, we make the distinction between strong stabilization, where the energy of every solution to (1.2) goes to 0 as $t \to \infty$ (or, equivalently, $e^{tG} \xrightarrow{t\to+\infty} 0$ strongly in $\mathcal{H}$) and uniform stabilization, where there is $f(t) \xrightarrow{t\to+\infty} 0$ such that for all solution u of (1.2), $E(u,t) \leqslant f(t)$ at all times $t \geqslant 0$. In this latter case, a standard argument shows that the decay is actually exponential. To determine whether stabilization holds involves necessarily additional assumptions of geometrical nature: for instance, it is clear that if the support of μ is localized on a nodal set of a Dirichlet eigenfunction of $-\Delta_g$, there are solutions of (1.2) with constant energy, and strong stabilization does not hold. On the other hand, for the standard damped wave equation – that is, with a smooth damping term of the form au_t with non-zero $a \in C^\infty(\Omega, \mathbb{R}_+)$, the geometric control hypothesis ensures a uniform decay [BLR92] following [RT74] for the case of empty boundary. In the present case of singular damping, our analysis does not allow to explore this question further than a spectral condition (Theorem 11 below). However, our main result give a natural condition for strong stabilization to hold:

Theorem 3. *Let μ be a finite positive measure on $\overline{\Omega}$ such that* (1.1) *holds. For $\lambda \in \mathrm{Spec}\,(-\Delta_g)$, denote by $\psi_\lambda \in L^2(\Omega)$ any eigenfunction with eigenvalue λ. The following statements are equivalent :*

(i) The semigroup e^{tG} satisfies $e^{tG} \xrightarrow{t\to+\infty} 0$ strongly.

(ii) For any $\lambda \in \mathrm{Spec}\,(-\Delta_g)$, and any eigenfunction $\psi_\lambda \in L^2(\Omega)$ with eigenvalue λ, the measure μ satisfies

$$\int_\Omega |\psi_\lambda|^2 d\mu > 0. \tag{1.3}$$

In other words, the energy of any solution of (1.2) *decays to 0 if and only if* (1.3) *holds true.*

Note that for a regular damping of the form $\mu = a(x)dV_g$ with non-zero $a \in C^\infty(\Omega, \mathbb{R}_+)$, condition *(ii)* is always satisfied since an eigenfunction cannot vanish on an open set by Holmgren's uniqueness theorem. We recover here a standard result, namely the strong stabilization for (non-trivial) smooth damping.

2. Measures that multiply $H^1_0(\Omega)$ to $H^{-1}(\Omega)$

In this section, let M be a compact, n-dimensional Riemannian manifold, and $\Omega \subset M$ a connected open subset. Let μ be a finite, positive measure on $\overline{\Omega}$. We want to give conditions that imply

$$M_\mu : H^1_0(\Omega) \longrightarrow H^{-1}(\Omega), \quad \text{compactly, where} \quad M_\mu f = f\mu. \tag{2.1}$$

It will actually be convenient to let μ be a positive finite measure on M, and ask when

$$M_\mu : H^1(M) \longrightarrow H^{-1}(M), \quad \text{compactly.} \tag{2.2}$$

Since $H^1_0(\Omega)$ is a closed linear subspace of $H^1(M)$, we always get (2.1) from (2.2), by restriction (also restricting μ to $\overline{\Omega}$). We emphasize that the support of μ can have nonempty intersection with $\partial\Omega$.

To start with, let us mention conditions for

$$M_\mu : H^1(M) \longrightarrow H^{-1}(M), \tag{2.3}$$

i.e., (2.2) but disregarding compactness. If we take $f \in H^1(M)$ and pair $M_\mu f$ with f, since μ is a positive measure, a necessary condition for (2.3) is that

$$\int_M |f|^2 \, d\mu \le C\|f\|^2_{H^1(M)}, \quad \forall\, f \in H^1(M). \tag{2.4}$$

Applying Cauchy's inequality to $\int_M f\overline{g}\, d\mu$, we see that (2.4) is also sufficient for (2.3). Furthermore, it follows from Theorem 1.2.2 of [MS09] that the existence of $A < \infty$ such that

$$\mu(S) \le A\,\mathrm{Cap}(S), \quad \forall\, \text{Borel } S \subset M \tag{2.5}$$

is necessary and sufficient for (2.4) to hold, hence for (2.3) to hold, where $\mathrm{Cap}(S)$ is a variant of electrostatic capacity, appropriate for this situation.

A general way to obtain compacity for the map (2.3) proceeds as follows. If $\dim M = n \ge 2$, standard results for pointwise multiplication in Sobolev spaces [RS96] ensure that

$$\begin{aligned} &f, g \in H^1(M) \Rightarrow fg \in H^{1,p}(M), \quad \text{for } p = \frac{n}{n-1} \ \text{ if } \ n \ge 3, \\ &f, g \in H^1(M) \Rightarrow fg \in H^{1,p}(M), \quad \text{for all } p < 2 \ \text{ if } \ n = 2. \end{aligned} \tag{2.6}$$

Since $C(M)$ is dense in the duals of such spaces $H^{1,p}(M)$, we deduce from the preceding equations that (2.2) holds under the following conditions.

$$\mu \in H^{1,n/(n-1)}(M)^* = H^{-1,n}(M), \quad \text{if} \quad n \geq 3, \tag{2.7}$$
$$\mu \in H^{-1,r}(M), \quad \text{for some} \quad r > 2, \quad \text{if} \quad n = 2.$$

2.1. **Measures related to hypersurfaces $S \subset M$.** We can use the above discussion to obtain the following class of singular measures satisfying (2.2).

Proposition 4. *Let a compact $S \subset M$ be locally a Lipschitz graph, of dimension $n-1$, equipped with surface measure, i.e., $(n-1)$-dimensional Haudorff measure, σ_S. Let*

$$\mu = h\,\sigma_S, \tag{2.8}$$

with

$$h \in L^{n-1}(S, d\sigma_S), \quad \textit{if} \quad n \geq 3, \tag{2.9}$$
$$h \in L^{1+\delta}(S, d\sigma_S), \quad \textit{for some } \delta > 0, \textit{ if } n = 2.$$

Then (2.7) *holds, hence* (2.2) *holds.*

Proof. In view of (2.6), this amounts to find conditions on $q' > 0$ so that

$$\int (fg) h\, d\sigma_S < \infty$$

when $h \in L^{q'}(S)$ and $fg \in H^{1,p}(M)$. If $n \geq 3$, we can apply the trace theorem, followed by the embedding theorem,

$$\mathrm{Tr} : H^{1,p}(M) \longrightarrow B^s_{p,p}(S) \subset L^q(S), \tag{2.10}$$

with

$$p = \frac{n}{n-1}, \quad s = 1 - \frac{1}{p}, \quad q = \frac{(n-1)p}{n-1-sp} = \frac{n-1}{n-2}, \quad q' = n-1. \tag{2.11}$$

For $n = 2$, (2.10) applies for all $p \in (1, 2)$, again with $s = 1 - 1/p$, and taking $p \nearrow 2$ yields $q \nearrow \infty$, hence $q' \searrow 1$. Thus (2.8)–(2.9) imply (2.7). □

Remark 5. The trace result is perhaps better known when S is smooth. (Cf. [BL76], Theorem 6.6.1.) However, all the function spaces involved are invariant under bi-Lipschitz maps.

Positive measures satisfying (2.2) can have much wilder support than a Lipschitz surface. For example, one can take an infinite sequence of measures μ_k satisfying the hypotheses of Proposition 4, supported on surfaces S_k, and set $\mu = \sum_{k=1}^{\infty} a_k \mu_k$, with positive a_k decreasing sufficiently fast.

Here is another class of examples. Let $\mathcal{O} \subset M$ be an open set whose boundary $\partial\mathcal{O}$ is locally the graph of a continuous function. Then one can take a smooth vector field X on M, vanishing nowhere on $\partial\mathcal{O}$, whose flow $\mathcal{F}^t_X$ has the property that, for each $y \in \partial\mathcal{O}$, $\mathcal{F}^t_X y$ belongs to $\mathcal{O}$ for small $t > 0$. Then

$$\mu = X\chi_{\mathcal{O}} \tag{2.12}$$

is a positive measure, supported by $\partial\mathcal{O}$, and it belongs to $H^{-1,\infty}(M)$. The positivity of μ is a consequence of the fact that

$$\chi_{\mathcal{O}} \circ \mathcal{F}_X^t \geq \chi_{\mathcal{O}}, \tag{2.13}$$

for all small $t > 0$, since

$$t^{-1}(\chi_{\mathcal{O}} \circ \mathcal{F}_X^t - \chi_{\mathcal{O}}) \longrightarrow X\chi_{\mathcal{O}} \ \text{ in } \ \mathcal{D}'(M), \ \text{ as } \ t \to 0. \tag{2.14}$$

In the last class of examples, (2.12), the support of μ has topological dimension $n-1$, but its Hausdorff dimension can be $> n-1$.

2.2. **Measures with fractal support.** We next produce measures satisfying (2.7) and supported on "fractal" sets of Hausdorff dimension $< n-1$. We make use of the following result, contained in Theorem 4.7.4 of [Zie89]. Here, $B_r(x)$ denotes the ball of radius r centered at x.

Lemma 6. *Let μ be a positive measure on M with the property that there exist $A < \infty$ and $\epsilon > 0$ such that*

$$\mu(B_r(x)) \leq Ar^{n-q+\epsilon}, \quad \forall\, r \in (0,1], \ x \in M. \tag{2.15}$$

Assume $q \in (1,n)$. Then

$$\mu \in H^{-1,p}(M), \quad p = q'. \tag{2.16}$$

We then see that (2.7) holds whenever

$$\mu(B_r(x)) \leq Ar^{\alpha}, \quad \alpha > n - 1 - \frac{1}{n-1}. \tag{2.17}$$

In particular, for $n = 2$, it suffices to have (2.17) for some $\alpha > 0$.

We will give some explicit examples of a compactly supported measure on $\mathbb{R}^2$ satisfying (2.17). It will be clear that many other examples can be constructed. We start with the Cantor middle third set $\mathcal{K} \subset [0,1]$. Now put $[0,1] \subset \mathbb{R} \subset \mathbb{R}^2$, say as part of the x-axis, so now $\mathcal{K} \subset \mathbb{R}^2$. As is well known (cf. [Tay06], p. 170), there is the α-dimensional Hausdorff measure computation

$$\mathcal{H}^{\alpha}(\mathcal{K}) = \gamma_{\alpha}, \quad \text{for } \ \alpha = \frac{\log 2}{\log 3} \approx 0.6309, \tag{2.18}$$

with $0 < \gamma_{\alpha} < \infty$ (in fact, $\gamma_{\alpha} = \pi^{\alpha/2}2^{-\alpha}/\Gamma(\alpha/2+1)$). Set

$$\mu = \mathcal{H}^{\alpha} \lfloor \mathcal{K}, \tag{2.19}$$

i.e., $\mu(S) = \mathcal{H}^{\alpha}(\mathcal{K} \cap S)$, for Borel sets $S \subset \mathbb{R}^2$. The self similarity of $\mathcal{K}$ enables one to show that

$$\mu(B_{3^{-k}}(x)) \leq C\, 2^{-k}, \tag{2.20}$$

which readily leads to (2.17), with α as in (2.18).

The Cantor middle third set described above is one of a family of Cantor sets $\mathcal{K}(\vartheta) \subset [0,1]$, defined for $\vartheta \in (0,1)$ as follows. Remove from $[0,1] = I$ the open interval of length $\vartheta\ell(I)$, with the same center as I, and repeat this process with the

other closed subintervals that remain. (Thus $\mathcal{K} = \mathcal{K}(1/3)$.) This time (cf. [Tay06], p. 171), one has

$$\mathcal{H}^\alpha(\mathcal{K}(\vartheta)) = \gamma_\alpha, \quad \alpha = \frac{\log 2}{\log b}, \quad b = \frac{2}{1-\vartheta}, \tag{2.21}$$

and again self-similarity yields

$$\mu(B_r(x)) \le Ar^\alpha, \tag{2.22}$$

with α as in (2.21), when

$$\mu = \mathcal{H}^\alpha \lfloor \mathcal{K}(\vartheta). \tag{2.23}$$

Note that

$$\vartheta \searrow 0 \Rightarrow \alpha \nearrow 1, \quad \vartheta \nearrow 1 \Rightarrow \alpha \searrow 0. \tag{2.24}$$

As before, we put $\mathcal{K}(\vartheta) \subset [0,1] \subset \mathcal{R} \subset \mathcal{R}^2$, and regard μ in (2.23) as a compactly supported measure on $\mathbb{R}^2$. Thus the push-forward of μ to a measure on a compact two-dimensional manifold M, via a locally bi-Lipschitz map, yields a measure on M satisfying (2.7), hence (2.2), whenever $0 < \vartheta < 1$.

One way to get measures on higher dimensional spaces satisfying (2.17) is to take products. Say $n = n_1 + n_2$ and μ_j are compactly supported measures on $\mathbb{R}^{n_j}$ satisfying

$$\mu_j(B_r(x_j)) \le C_j r^{\alpha_j}, \quad j = 1, 2, \quad x_j \in \mathbb{R}^{n_j}. \tag{2.25}$$

If $x = (x_1, x_2) \in \mathbb{R}^n$, note that $B_r(x) \subset B_r(x_1) \times B_r(x_2)$, so if

$$\mu = \mu_1 \times \mu_2 \tag{2.26}$$

is the product measure on $\mathbb{R}^n$, we have

$$\mu(B_r(x)) \le \mu_1(B_r(x_1))\mu_2(B_r(x_2)) \le C_1 C_2 r^{\alpha_1+\alpha_2}, \quad x \in \mathbb{R}^n. \tag{2.27}$$

If $\alpha = \alpha_1 + \alpha_2$ satisfies the condition on α in (2.17), we get a compactly supported measure on $\mathbb{R}^n$ whose push-forward to a measure on an n-dimensional compact manifold M, via a locally bi-Lipschitz map, satisfies (2.17), hence (2.7), hence (2.2).

For example, take $\vartheta_j \in (0,1)$, and set

$$\mu_j = \mathcal{H}^{\alpha_j} \lfloor \mathcal{K}(\vartheta_j), \tag{2.28}$$

with α_j as in (2.21), i.e.,

$$\alpha_j = \frac{\log 2}{\log b_j}, \quad b_j = \frac{2}{1-\vartheta_j}. \tag{2.29}$$

We regard μ_1 as a measure on $\mathbb{R}^2$ and μ_2 as a measure on $\mathbb{R}$, via $\mathcal{K}(\vartheta_1) \subset [0,1] \subset \mathbb{R} \subset \mathbb{R}^2$ and $\mathcal{K}(\vartheta_2) \subset [0,1] \subset \mathbb{R}$. Thus $\mu = \mu_1 \times \mu_2$ is a compactly supported measure on $\mathbb{R}^3$ (actually supported on a 2D linear subspace of $\mathbb{R}^3$). In this case, the condition for (2.21) to hold is

$$\alpha_1 + \alpha_2 > \frac{3}{2}. \tag{2.30}$$

Looking at (2.18), we see that (2.30) fails when $\vartheta_1 = \vartheta_2 = 1/3$. In case $\vartheta_1 = \vartheta_2 = \vartheta$, the condition that (2.30) hold is that

$$2\,\frac{\log 2}{\log 2 - \log(1-\vartheta)} > \frac{3}{2}, \tag{2.31}$$

or equivalently,

$$\vartheta < 1 - 2^{-1/3} \approx 0.2063. \tag{2.32}$$

Remark 7. When the measures μ_j are given by (2.28), $\mu = \mu_1 \times \mu_2$ is supported on $\mathcal{K}(\vartheta_1) \times \mathcal{K}(\vartheta_2)$, but it is generally *not* $(\alpha_1 + \alpha_2)$-dimensional Hausdorff measure on this set. See pp. 70–74 of [Fal86] for a discussion of this matter.

Returning to generalities, we mention that while (2.15) is a sufficient condition for (2.16), it is not quite necessary. There is a (somewhat more elaborate) necessary and sufficient condition for (2.16) to hold, provided $1 < q < n$, given in terms of an estimate on $\mu(B_r(x))$. See Theorem 4.7.5 of [Zie89]. The condition is subtly weaker than (2.15).

3. Proof of theorem 1

As usual, when dealing with the wave equation, it behooves us to write the second order (in time) scalar equation as a first order system. Recall the vector space $\mathcal{H} = L^2(\Omega) \oplus H_0^1(\Omega)$. The inner product on $\mathcal{H}$ is defined as follows: if $U = (v, w)$ and $F = (f, g)$ are in $\mathcal{H}$, then

$$\begin{aligned} \langle U, F \rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \langle v, f \rangle_{L^2(\Omega)} + \langle \nabla w, \nabla g \rangle_{L^2(\Omega)}. \end{aligned}$$

Again, since we are using Dirichlet boundary conditions, the homogeneous H_0^1 inner product induces a norm, so the inner product on $\mathcal{H}$ induces a norm on $\mathcal{H}$.

Now suppose $u \in C((0,\infty)_t, H_0^1(\Omega))$ solves the damped wave equation (1.2), and set

$$U = \begin{pmatrix} u_t \\ u \end{pmatrix}.$$

Differentiating, we have

$$\begin{aligned} U_t &= \begin{pmatrix} u_{tt} \\ u_t \end{pmatrix} \\ &= \begin{pmatrix} -M u_t + \Delta_g u \\ u_t \end{pmatrix} \\ &= \begin{pmatrix} -M & \Delta_g \\ I & 0 \end{pmatrix} \begin{pmatrix} u_t \\ u \end{pmatrix} = GU, \end{aligned}$$

where G is the matrix operator

$$G = \begin{pmatrix} -M & \Delta_g \\ I & 0 \end{pmatrix}. \tag{3.1}$$

Let us compute the weak formulation of the equation $U_t = GU$. If $\Psi = (\varphi, \psi) \in \mathcal{H} \cap D(G)$, where $D(G)$ is the domain of G (to be computed later), then

$$\begin{aligned}\langle U_t, \Psi\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} u_{tt} \\ u_t \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle \begin{pmatrix} -Mu_t + \Delta_g u \\ u_t \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -\langle Mu_t, \varphi\rangle - \langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)} + \langle \nabla u_t, \nabla\psi\rangle_{L^2(\Omega)}.\end{aligned}$$

The left hand side is similarly computed

$$\left\langle \begin{pmatrix} u_{tt} \\ u_t \end{pmatrix}, \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\rangle = \langle u_{tt}, \varphi\rangle_{L^2(\Omega)} + \langle \nabla u_t, \nabla\psi\rangle_{L^2(\Omega)}.$$

Making the obvious cancellations, we have

$$\langle u_{tt}, \varphi\rangle_{L^2(\Omega)} = -\langle Mu_t, \varphi\rangle - \langle \nabla u, \nabla\varphi\rangle_{L^2(\Omega)},$$

which is the same weak formulation as the scalar equation. Putting the other terms back in, we have the weak formulation in terms of the system:

$$\langle U_t, \Psi\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} -\nabla u \\ u_t \end{pmatrix}, \begin{pmatrix} \nabla\varphi \\ \psi \end{pmatrix} \right\rangle_{\mathcal{H}} - \langle Mu_t, \varphi\rangle.$$

We begin by computing the adjoint of the matrix operator G. Let

$$U = \begin{pmatrix} v \\ w \end{pmatrix}, F = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H} \cap D(G),$$

where the domain $D(G)$ will be determined shortly, and compute

$$\begin{aligned}\langle GU, F\rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} -M & \Delta_g \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle \begin{pmatrix} -Mv + \Delta w \\ v \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= -\langle Mv, f\rangle + \langle \Delta w, f\rangle_{L^2(\Omega)} + \langle \nabla v, \nabla g\rangle_{L^2(\Omega)} \\ &= \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} -Mf - \Delta_g g \\ -f \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \left\langle \begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} -M & -\Delta_g \\ -I & 0 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}},\end{aligned}$$

so that the adjoint is

$$G^* = \begin{pmatrix} -M & -\Delta_g \\ -I & 0 \end{pmatrix}.$$

Since $G \neq G^*$, we will consider only the real part to show that G is dissipative. That is, we need to show $\operatorname{Re}\langle GU, U\rangle \leqslant 0$. Let

$$U = \begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{H} \cap D(G).$$

We compute

$$\begin{aligned}
\operatorname{Re}\langle GU, U\rangle_{\mathcal{H}} &= \operatorname{Re}\left\langle \begin{pmatrix} -M & \Delta_g \\ I & 0 \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right\rangle_{\mathcal{H}} \\
&= \operatorname{Re}\left\langle \begin{pmatrix} -Mv + \Delta w \\ v \end{pmatrix}, \begin{pmatrix} v \\ w \end{pmatrix}\right\rangle_{\mathcal{H}} \\
&= -\langle Mv, v\rangle + \operatorname{Re}\langle \Delta_g w, v\rangle_{L^2(\Omega)} + \operatorname{Re}\langle \nabla v, \nabla w\rangle_{L^2(\Omega)} \\
&= -\langle Mv, v\rangle - \operatorname{Re}\langle \nabla w, \nabla v\rangle_{L^2(\Omega)} + \operatorname{Re}\langle \nabla v, \nabla w\rangle_{L^2(\Omega)} \\
&= -\langle Mv, v\rangle \\
&\leqslant 0,
\end{aligned}$$

since M is symmetric and non-negative according to Assumption 1. This shows that G is dissipative.

In order to conclude that G is maximal dissipative, we also need to show that $(I - G)$ is an isomorphism between appropriate spaces. In particular, we will describe the domain $D(I - G)$ and show that

$$(I - G) : D(I - G) \to \mathcal{H}$$

is an isomorphism. We will also conclude that the inverse mapping $(I - G)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. This is where we use the assumed compact mapping properties of $M : H_0^1(\Omega) \to H^{-1}(\Omega)$.

Let

$$U = \begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{H} \cap D(G),$$

and

$$F = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{H}.$$

We want to understand the solvability of $(I - G)U = F$ and show that for any $F \in \mathcal{H}$, there is a unique $U \in D(I - G)$ solving this equation. Writing out in components:

$$\begin{aligned}
(I - G)\begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} (1 + M)v - \Delta_g w \\ -v + w \end{pmatrix} \\
&= \begin{pmatrix} f \\ g \end{pmatrix},
\end{aligned}$$

or

$$\begin{cases} (1 + M)v - \Delta_g w = f, \\ -v + w = g. \end{cases}$$

Using the second equation to eliminate $v = w - g$ from the first, we get the scalar equation

$$(1 + M)w - \Delta_g w = f + (1 + M)g. \tag{3.2}$$

Now $F \in \mathcal{H}$ means that $f \in L^2(\Omega)$ and $g \in H_0^1(\Omega)$. The assumed mapping properties of M implies

$$Mg \in H^{-1}(\Omega).$$

Hence the right hand side of (3.2) is in $H^{-1}(\Omega)$ or better. Since $U \in D(I-G) \subset \mathcal{H}$, we know that $w \in H_0^1(\Omega)$ or better.

Now observe that $(1 - \Delta_g) : H_0^1(\Omega) \to H^{-1}(\Omega)$ is an isomorphism by the usual Fredholm theory. Since $M : H_0^1(\Omega) \to H^{-1}(\Omega)$ is compact,

$$(1 - \Delta_g + M) : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is a compact perturbation of an isomorphism, hence has Fredholm index 0. We therefore know that the codimension of the image is the dimension of the kernel. We therefore need only establish injectivity to recover surjectivity. Assume then that $w \in \ker(1 - \Delta_g + M)$ and compute

$$\begin{aligned} 0 &= \langle (1 - \Delta_g + M)w, w \rangle_{L^2(\Omega)} \\ &= \|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)} + \langle Mw, w \rangle \\ &\geqslant \|w\|^2_{L^2(\Omega)} + \|\nabla w\|^2_{L^2(\Omega)} \end{aligned}$$

since M is symmetric and non-negative. Hence $w = 0$ and the kernel is trivial as claimed.

We have shown that

$$(1 - \Delta_g + M) : H_0^1(\Omega) \to H^{-1}(\Omega)$$

is an isomorphism. We will use this, together with the mapping properties of M to establish that $(I - G)^{-1}$ is compact and also describe $D(I - G)$. We give the right hand side of (3.2) a name. Let $\Phi : \mathcal{H} \to H^{-1}$ be defined by

$$\Phi \begin{pmatrix} f \\ g \end{pmatrix} = f + g + Mg.$$

Recall that $f \in L^2(\Omega)$ and $g \in H_0^1(\Omega)$. The embeddings

$$L^2(\Omega) \hookrightarrow H^{-1}(\Omega), \quad H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$$

are compact, as is $M : H_0^1(\Omega) \to H^{-1}(\Omega)$, hence $\Phi : \mathcal{H} \to H^{-1}$ is compact. Returning to (3.2), we know the operator on the left hand side is an isomorphism, hence invertible as a bounded linear operator. We have

$$w = (1 - \Delta_g + M)^{-1} \Phi \begin{pmatrix} f \\ g \end{pmatrix} =: \Psi \begin{pmatrix} f \\ g \end{pmatrix}.$$

The mapping Ψ defined above is a composition of a bounded linear operator with a compact operator, taking values in $H_0^1(\Omega)$. Hence $\Psi : \mathcal{H} \to H_0^1(\Omega)$ is compact. Recalling that $v = w - g$, we have

$$v = \Psi \begin{pmatrix} f \\ g \end{pmatrix} - g.$$

The original equation we considered was $(I - G)U = F$. In particular, the second component in F, $g \in H_0^1(\Omega)$. Hence

$$v = \Psi \begin{pmatrix} f \\ g \end{pmatrix} - g \in H_0^1(\Omega)$$

if $F \in \mathcal{H}$. As the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the mapping defining v is compact when mapping into $L^2(\Omega)$. Taken together, we write for U so obtained

$$U = (I - G)^{-1} F,$$

where

$$(I - G)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Psi \begin{pmatrix} f \\ g \end{pmatrix} - g \\ \Psi \begin{pmatrix} f \\ g \end{pmatrix} \end{pmatrix}.$$

Then as a mapping

$$(I - G)^{-1} : \mathcal{H} \to \mathcal{H},$$

$(I - G)^{-1}$ is compact. The domain $D(I - G)$ is the image of $\mathcal{H}$ under this transformation.

4. Decay

In this section, we examine the general spectral theory of G and the associated semigroup e^{tG} which is the solution operator for the damped wave system. We have shown that $(I - G)^{-1} : \mathcal{H} \to \mathcal{H}$ is compact. It follows that G is closed and $1 \notin \operatorname{Spec} G$. As a classical result (see [Kat95], Theorem III.6.29) we deduce that G has compact resolvent with eigenvalues of finite multiplicities, accumulating only possibly at $\infty \in \overline{\mathbb{C}}$. As a direct consequence of $\operatorname{Re} \langle GU, U \rangle_{\mathcal{H}} \leqslant 0$ for any $U \in \mathcal{H}$, the spectrum of G satisfies

$$\operatorname{Spec} G \subset \{z \in \mathbb{C},\ \operatorname{Re} z \leqslant 0\},$$

so the associated semigroup

$$e^{tG} : D(I - G) \to \mathcal{H}$$

is a contraction, in the sense that

$$\|e^{tG}\|_{\mathcal{H} \to \mathcal{H}} \leqslant 1.$$

We also deduce that in particular, the image of $\mathcal{H}$ is $D(I - G)$, which is a subspace of $H_0^1 \oplus H_0^1 \subset \mathcal{H}$. When operating on eigenfunctions of G, the evolution is easy to compute. If

$$F \in D(I - G)$$

is an eigenfunction of G with eigenvalue $\zeta \in \mathbb{C}$, we have $GF = \zeta F$ and then

$$e^{tG} F = e^{t\zeta} F.$$

Hence the decay properties of the semigroup are dictated by the eigenvalues of G. In particular, since ζ must lie in the closed left half-plane in $\mathbb{C}$, any purely imaginary eigenvalues correspond to non-decaying states. In the absence of any

purely imaginary eigenvalues, the asymptotic distribution of $\operatorname{Re}\zeta$ as $|\operatorname{Im}\zeta| \to \infty$ determines the decay rate (see, for example, [Chr09]).

Let us fix some notations. If $\zeta \in \operatorname{Spec}(G)$ is an eigenvalue, set $V_\zeta \subset D(G)$ to be the generalized eigenspace corresponding to ζ, and let $V_\zeta^\# \subset V_\zeta$ be the primitive eigenspace. That is, $V_\zeta^\#$ consists of eigenvectors of G with eigenvalue ζ, and V_ζ consists of eigenvectors of G^k, $k \geqslant 1$, with eigenvalue ζ. As we noted above, each V_ζ and $V_\zeta^\#$ is finite dimensional.

Let $\zeta \in \operatorname{Spec} G$ so $\operatorname{Re}\zeta \leqslant 0$. Take

$$\begin{pmatrix} f \\ g \end{pmatrix} \in V_\zeta^\#.$$

Recall that this implies

$$g \in H_0^1, \qquad f = \zeta g, \qquad \Delta g = \zeta^2 g + \zeta M g.$$

Proposition 8. *If $(f,g) \in V_\zeta^\#$ with $\|(f,g)\|_{\mathcal{H}} = 1$ then $\operatorname{Re}\zeta = -\langle Mf, f\rangle$.*

In particular, if M is the multiplication by the measure μ, then

$$\operatorname{Re}\zeta = -\int_\Omega |f|^2 d\mu.$$

Proof. Note that

$$G\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \Delta g - Mf \\ f \end{pmatrix} = \zeta \begin{pmatrix} f \\ g \end{pmatrix},$$

so

$$\left\langle G\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle \Delta g, f\rangle_{L^2(\Omega)} - \langle Mf, f\rangle - \langle \Delta f, g\rangle_{L^2(\Omega)}.$$

Then the above implies

$$\begin{aligned} \zeta \left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|_{\mathcal{H}}^2 &= \langle \Delta g, f\rangle_{L^2(\Omega)} - \langle Mf, f\rangle - \overline{\langle \zeta f + Mf, f\rangle}_{L^2(\Omega)} \\ &= \zeta \|f\|_{L^2}^2 + \langle Mf, f\rangle - \langle Mf, f\rangle - \overline{\zeta}\|f\|_{L^2}^2 - \langle Mf, f\rangle \\ &= (\zeta - \overline{\zeta})\|f\|_{L^2}^2 - \langle Mf, f\rangle. \end{aligned}$$

Taking the real part gives the result. □

4.1. **Primitive eigenspaces with purely imaginary eigenvalues.** Let us denote the primitive eigenspaces with purely imaginary eigenvalues by $V_{i\lambda}^\#$, where this notation assumes $\lambda \in \mathbb{R}$. An immediate consequence of the preceding proposition is

Corollary 9. *We have*

$$\begin{pmatrix} v \\ w \end{pmatrix} \in V_{i\lambda}^\# \quad \Leftrightarrow \quad \langle Mv, v\rangle = 0.$$

In particular, if $M = M_\mu$ is the multiplication by a non-negative measure satisfying the compactness assumptions, then a necessary condition for $(v, w)^T \in V_{i\lambda}^{\#}$ is that

$$\langle M_\mu v, v\rangle = \int_\Omega |v|^2 d\mu = 0.$$

This is true if and only if v is supported in the 0-set of the measure μ. A similar converse holds as well. If v satisfies $\int_\Omega |v|^2 d\mu = 0$, then, we recall that $M_\mu v \in H^{-1}(\Omega)$, so for any $f \in H_0^1(\Omega)$, we have

$$\begin{aligned} |\langle M_\mu v, f\rangle| &= \left| \int_\Omega v \bar{f} d\mu \right| \\ &\leqslant \left(\int_\Omega |v|^2 d\mu \right)^{1/2} \left(\int_\Omega |f|^2 d\mu \right)^{1/2}. \end{aligned}$$

The latter integral is bounded, so the use of Hölder's inequality is justified; the first integral is 0 by assumption. Hence we have that $M_\mu v = 0$ in $H^{-1}(\Omega)$ (that is, in the weak sense).

Returning to the matrix

$$G : D(G) \to H^{-1}(\Omega) \oplus H_0^1(\Omega),$$

we have if $(v, w)^T \in V_{i\lambda}^{\#}$, then

$$\begin{aligned} G \begin{pmatrix} v \\ w \end{pmatrix} &= \begin{pmatrix} -M_\mu v + \Delta w \\ v \end{pmatrix} \\ &= \begin{pmatrix} \Delta w \\ v \end{pmatrix} \\ &= \begin{pmatrix} 0 & \Delta \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \end{aligned}$$

But

$$G \begin{pmatrix} v \\ w \end{pmatrix} = i\lambda \begin{pmatrix} v \\ w \end{pmatrix},$$

so

$$\begin{cases} \Delta w = i\lambda v \\ v = i\lambda w, \end{cases}$$

Substituting, we have $-\Delta w = \lambda^2 w$, which means that w is a Dirichlet eigenfunction on Ω. Hence if w is such an eigenfunction and $v = i\lambda w$, then the solution u to the damped wave equation

$$\begin{cases} (\partial_t^2 - \Delta + M_\mu \partial_t) u = 0, \text{ on } (0, \infty)_t \times \Omega, \\ u|_{\partial\Omega} = 0, \text{ on } [0, \infty)_t \times \partial\Omega, \\ u(0, x) = w(x), \ u_t(0, x) = v(x), \end{cases}$$

satisfies the usual undamped wave equation:

$$\begin{cases} (\partial_t^2 - \Delta)u = 0, \text{ on } (0,\infty)_t \times \Omega, \\ u|_{\partial\Omega} = 0, \text{ on } [0,\infty)_t \times \partial\Omega, \\ u(0,x) = w(x), \ u_t(0,x) = v(x). \end{cases}$$

Next, let

$$\mathcal{M} = \overline{\bigoplus_{\lambda\in\mathbb{R}} V_{i\lambda}^{\#}}$$

denote the $\mathcal{H}$ closure of all of the eigenspaces with purely imaginary eigenvalue. Since the spectrum is pure point, this is, of course, a sum over at most countably many such $\lambda \in \mathbb{R}$. Let $\mathcal{M}^\perp \subset \mathcal{H}$ be the orthogonal complement to $\mathcal{M}$. The following proposition is the central result of this paper, it will readily imply Theorem 3.

Proposition 10. *Let G, $\mathcal{M}$, and $\mathcal{M}^\perp$ be defined as above.*

1. *For each t, the semigroup $e^{tG} : \mathcal{M} \to \mathcal{M}$ is unitary.*
2. *The space $\mathcal{M}^\perp$ is also invariant under e^{tG}.*
3. *The resolvent $(I-G)^{-1} : \mathcal{M}^\perp \to \mathcal{M}^\perp$ is compact.*
4. *If $F \in \mathcal{M}^\perp$, then*
$$\|e^{tG}F\|_{\mathcal{H}} \to 0,$$
as $t \to \infty$.

Proof. Part 1 follows directly from the definition of $\mathcal{M}$. To prove part 2, it suffices to show that no generalized eigenvector of G with imaginary eigenvalue is in $\mathcal{M}^\perp$. Suppose for some $\lambda \in \mathbb{R}$ that

$$\mathcal{M}^\perp \cap V_{i\lambda} \neq \emptyset.$$

Since $V_{i\lambda}$ is finite dimensional, the vector space $V = \mathcal{M}^\perp \cap V_{i\lambda}$ is finite dimensional, and consists of generalized eigenvectors of G. That means

$$G|_V = A$$

is a finite square matrix A. Putting A into Jordan canonical form shows us that there must be at least one honest eigenvector of G in V, still with eigenvalue $i\lambda$. Hence $V \cap V_{i\lambda}^{\#} \neq \emptyset$, which is a contradiction. This shows that for every $i\lambda \in \sigma(G)$, $V_{i\lambda} \subset \mathcal{M}$, which proves 2. To prove part 3, we write the resolvent as an integral:

$$(I-G)^{-1} = \int_0^\infty e^{-t}e^{tG}dt.$$

Since e^{tG} is a contraction semigroup, the integral converges in operator norm, and indeed we can formally integrate by parts to derive this formula. Since $e^{tG} : \mathcal{M}^\perp \to \mathcal{M}^\perp$, we have that the resolvent

$$(I-G)^{-1} : \mathcal{M}^\perp \to \mathcal{M}^\perp$$

as well, and is compact, since it is compact on the larger space $\mathcal{H}$.

For the last part, we use the following abstract decay result [BRT82]: if A is a maximal dissipative operator on a Hilbert space such that A has compact resolvent and no purely imaginary eigenvalues, then

$$\lim_{t\to+\infty} e^{tA} = 0, \quad \text{strongly.}$$

The conclusion of 4 follows by considering $e^{tG} : \mathcal{M}^\perp \to \mathcal{M}^\perp$, since we have seen above that $G|_{\mathcal{M}^\perp}$ has no purely imaginary eigenvalues. □

As an immediate corollary of the above result, note that $\lim_{t\to+\infty} e^{tA} = 0$ strongly if and only if $\mathcal{M} = \emptyset$. To get Theorem 3, let $M = M_\mu$ as in Section 2. From the discussion following Corollary 9, $\mathcal{M} \neq \emptyset$ if and only if there is a Dirichlet eigenfunction ψ_λ of $-\Delta_g$ such that

$$\int |\psi_\lambda| d\mu = 0,$$

in which case ψ_λ is a stationary solution of (1.2) and $\psi_\lambda \in \mathcal{M}$.

4.2. **Spectral gap and exponential decay.** We say that the semigroup $(e^{tG})_{t\geqslant 0}$ decays exponentially if there is $C, \beta > 0$ such that

$$\|e^{tG}\|_{\mathcal{L}(\mathcal{H})} \leqslant Ce^{-\beta t}, \qquad \forall t \geqslant 0.$$

Hence uniform stabilization is equivalent to an exponential decay of the semigroup. On the other hand, we say that G has a spectral gap if there is $A > 0$ such that $\forall \zeta \in \operatorname{Spec} G$,

$$|\operatorname{Re}\zeta| \geqslant A.$$

Exponential decay of the semigroup and spectral gap are not necessarily equivalent in non-selfadjoint spectral problems, due to pseudospectral phenomena. However we have the following result :

Theorem 11. *Let $A > 0$ and consider the strip $\mathcal{S} = \{z \in \mathbb{C} : \operatorname{Re} z \in [-A, 0]\}$.*

(i) If the operator G has no spectral gap, then

$$\|e^{tG}\|_{\mathcal{L}(\mathcal{H})} = 1, \quad \forall t \geqslant 0.$$

(ii) Suppose that for some $A > 0$ the strip $\mathcal{S}$ is in the resolvent set of G, and that

$$\sup_{z\in\mathcal{S}} \|(z - G)^{-1}\| < \infty.$$

Then the semigroup $(e^{tG})_{t\geqslant 0}$ decays exponentially.

Proof. It is clear that if there exists $(\zeta_j)_{j\in\mathbb{N}}$ with $\zeta_j \in \operatorname{Spec} G$ and $\operatorname{Re}\zeta_j \xrightarrow{j\to\infty} 0$, then $\|e^{tG}\|_{\mathcal{L}(\mathcal{H})} = 1, \quad \forall t \geqslant 0$.

Conversely, let us assume that there is $A > 0$ such that $\forall \zeta \in \operatorname{Spec} G$, $|\operatorname{Re}\zeta| \geqslant A$. The preimage of $\mathcal{C} = \{z \in \mathbb{C} : e^{-A} \leqslant |z| \leqslant 1\}$ by the exponential map is precisely $\mathcal{S}$, and the uniform boundedness of the resolvent on $\mathcal{S}$ allows us to apply

a theorem of Prüss [Prü84] to conclude that $\mathcal{C}$ is in the resolvent set of e^G. Hence the spectral radius of e^G is $\leqslant e^{-A}$, namely

$$\limsup_{k\to\infty} \|e^{kG}\|_{\mathcal{L}(\mathcal{H})}^{\frac{1}{k}} \leqslant e^{-A} < 1.$$

This implies that $\|e^{tG}\|_{\mathcal{L}(\mathcal{H})} < 1$ for some $t > 0$ and concludes the proof of the theorem by using the semigroup property. □

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