SMALL-SCALE MASS ESTIMATES FOR LAPLACE EIGENFUNCTIONS ON COMPACT C^2 MANIFOLDS WITH BOUNDARY

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ABSTRACT. Let Ω be an n-dimensional compact Riemannian manifold with C^2 boundary, and consider L^2-normalized eigenfunctions $-\Delta\phi_\lambda = \lambda^2\phi_\lambda$ with Dirichlet or Neumann boundary conditions .

In this note, we extend well-known interior nonconcentration bounds up to the boundary. Specifically, in Theorem 1 using purely stationary, local methods, we prove that for such Ω, it follows that for *any* $x_0 \in \overline{\Omega}$ (including boundary points) and for all $\mu \geq h$,

$$\|\phi_\lambda\|^2_{B(x_0,\mu)\cap\Omega} = O(\mu). \tag{1}$$

In Theorem 2 we extend a result of Sogge [So] to manifolds with smooth boundary and show that

$$\|\phi_\lambda\|_{L^\infty(\Omega)} \leq C\lambda^{\frac{n}{2}} \cdot \sup_{x\in\Omega} \|\phi_\lambda\|_{L^2(B(x,\lambda^{-1})\cap\Omega)}. \tag{2}$$

The sharp sup bounds $\|\phi_\lambda\|_{L^\infty(\Omega)} = O(\lambda^{\frac{n-1}{2}})$ first proved by Grieser in [Gr] are then an immediate consequence of Theorems 1 and 2.

1. INTRODUCTION

A fundamental issue regarding eigenfunctions involves their concentration properties on small balls with radius that depends on the eigenvalue λ^2 as $\lambda \to \infty$. As pointed out in [So], if $\partial\Omega = \emptyset$, using the explicit asymptotic formula for the half-wave operator $e^{it\sqrt{-\Delta}} : C^\infty(\Omega) \to C^\infty(\Omega)$ it is not hard to prove that there exists $C_\Omega > 0$ such that

$$\|\phi_\lambda\|^2_{L^2(B(r))} = O(r)\|\phi_\lambda\|^2_{L^2(\Omega)}, \quad \forall r \geq C_\Omega\lambda^{-1} \tag{3}$$

We refer to estimates of the form (3) as *non-concentration* bounds. The example of highest weight spherical harmonics on the round sphere (see Remark 1 below) shows that (3) is, in general, sharp. However, in certain cases, one expects improvements. For instance, in the case of surfaces with non-positive curvature, one can get logarithmic improvements [So, Han].

Since the wave parametrix on manifolds with boundary is quite complicated, it is of interest to give proof of (3) using purely stationary methods. The first result of this paper in Theorem 1 an extension of the bounds in (3) to Dirichlet or Neumann eigenfunctions in the case the Ω is a compact manifold with C^2 boundary. To avoid the usual complications arising from the complicated behaviour of the wave parametrix near the boundary, our basic arguments here use a stationary Rellich commutator argument and avoid using the wave propagator altogether.

It is useful at this point to switch to the convenient semiclassical scaling $h = \lambda^{-1}$ and consider eigenfunctions $-h^2\Delta\phi_h = \phi_h$.

Theorem 1. *Let (Ω^n, g) be a compact, Riemannian manifold with C^2 boundary. Then, for any $p_0 \in \overline{\Omega}$ and Laplace eigenfunction ϕ_h with eigenvalue h^{-2}, it follows that*

$$\|\phi_h\|^2_{L^2(B(p_0,\mu)\cap\overline{\Omega})} = O(\mu); \quad \mu \geq h. \tag{4}$$

Remark 1. The estimate in Theorem 4 is sharp for all $\mu \geq h^{1/2}$. To see this, one need only consider eigenfunctions that are Gaussian beams. For concreteness, let (S^2, g) be the round sphere. In terms of Euclidean coordinates $(x, y, z) \in \mathbb{R}^3$, the highest-weight spherical harmonics on $S^2 = \{(x, y, z) \in \mathbb{R}^3, x^2 + y^2 + z^2 = 1\}$ are given by

$$\phi_h(x, y, z) = (2\pi n)^{1/4}(x + iy)^n; \quad n = 1, 2, 3, ...$$

Setting $h = n^{-1}$ and noting that $|\phi_h(x, y, z)| = (2\pi h)^{-1/4}(1-z^2)^{1/h}$ when $(x, y, z) \in S^2$,

Assuming $\mu \geq h^{1/2}$, it follows that for any $p \in \{(x, y, z) \in S^2, z = 0\}$,

$$\|\phi_h\|^2_{B(p,\mu)} \approx C_1 h^{-1/2}\mu \int_{|z|<\mu} (1 - z^2)^{2/h}\, dz$$

$$\approx C_2\mu h^{-1/2} \int_{|z|<\mu} e^{-z^2/h}\, dz \approx \mu \int_{|w|<\mu h^{-1/2}} e^{-w^2} dw \approx \mu,$$

and so, the non-concentration bounds are sharp when $\mu \in [h^{1/2}, 1]$.

In the special case where the ϕ_h satisfy polynomial small-scale quantum ergodicity (SSQE) on a scales $\mu \geq h$ since the volume of a ball of radius μ is μ^n one putatively expects a bound of $O(\mu^n)$ on the RHS in Theorem 1. Unfortunately, to our knowledge, there are no rigorous results on polynomial SSQE and seem well out of reach at present. Logarithmic SSQE was proved by X. Han [Han].

Remark 2. In a companion paper [CT], using a different Rellich-type commutator argument, we prove the standard non-concentration bounds in Theorem 1 in the special case of bounded, *convex* Euclidean planar domains with corners under the constraint that $\mu(h) \geq h^{1-\epsilon}$ for any fixed $\epsilon > 0$.

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2. Proof of Theorem 1

Proof. For simplicity, we will assume throughout the proof that the eigenfunctions ϕ_h are real-valued, however the general case can be obtained by taking real/imaginary parts where necessary.

2.1. **Interior estimates on scales $\mu \geq h$.** We first give a stationary proof of the non-concentration result in the interior of Ω using an h-microlocal Rellich commutator argument. Alternatively, a modified wave-parametrix argument in [So] also holds near interior points. Extension of wave methods to prove non-concentration bounds to manifolds with boundary is substantially more subtle due to the complicated nature of the wave operator near the glancing set. Consequently, it of interest to pursue stationary

methods. In this subsection, we begin with the simplest case of interior estimates. We assume here that (Ω, g) is an n-dimensional Riemannian manifold (with or without boundary) and since we are only interested in interior estimates here, this additional generality causes no additional complication. However, as we discuss in more detail below, the extension of non-concentration estimates to boundaries is substantially more involved.

Fix an interior point $z_0 \in \mathring{\Omega}$ and set $P(h) := -h^2\Delta_g - 1$ with principal symbol $p(x,\xi) = |\xi|_g^2 - 1$, where $|\xi|_g^2 = \sum_{i,j=1}^n g^{ij}(x)\xi_i\xi_j$. First, by energy concentration of eigenfunctions ϕ_h with $P(h)\phi_h = 0$, one can microlocalize to a fixed compact neighbourhood of the characteristic variety $S^*M = p^{-1}(0)$. More precisely, given any open interior neighbourhood $U \ni z_0$ and for fixed $\epsilon > 0$, we set $S^*_\varepsilon = \{\xi \in T^*_xM; |1 - |\xi|_g^2| \leq \epsilon.\}$ Then given any cutoff $\psi \in C_0^\infty(S^*_\epsilon)$ with the property that $\psi|_{S^*_{\varepsilon/2}} = 1$, by well-known concentration estimates [Zw]

$$\|(1 - \psi(x, hD))\phi_h\|_{C^k(U)} = O_k(h^\infty).$$

Then, given any cutoff $\rho \in C_0^\infty(U)$, we set

$$v_h := \rho(x)\psi(hD)\phi_h \in C_0^\infty(U)$$

and note that since $P(h)\phi_h = 0$ and $\rho(x)\psi(hD) \in Op_h(S^0)$,

$$P(x, hD)v_h = [P(x, hD), \rho(x)\psi(hD)]\phi_h = O_{L^2}(h) \tag{5}$$

by L^2-boundedness.

In the following, it suffices to work with the microlocalized eigenfunctions $v_h := \psi(x, hD)\phi_h$. Since the real principal type condition $d_\xi\, p(x,\xi) \neq 0$ is satisfied for $(x,\xi) \in S^*_\varepsilon U$, it follows by an application of the implicit function theorem to $p(x,\xi)$ that there are points $w_j \in T^*U; j = 1, .., N$ with open neighbourhoods $V_{w_j}; j = 1, ...N$, such that

$$S^*_\varepsilon U \subset \bigcup_{j=1}^N V_{w_j}$$

and with fiber coordinates $\xi = (\xi_k, \xi')$ on V_{w_j},

$$p(x,\xi) = e_j(x,\xi)\big(\xi_k - a_j(x,\xi')\big); \quad e_j(x,\xi) \geq C_j > 0, \ \ (x,\xi) \in V_{w_j}. \tag{6}$$

Let $\rho_j \in C_0^\infty(T^*U); j = 1, ..., N$ be a partition of unity subordinate to the cover $\{V_{w_j}\}_{j=1}^N$. It follows from (6) and L^2-boundedness that with another cutoff $\tilde{\rho_j} \ni \rho_j$ and $\tilde{\rho_j} \in C_0^\infty(V_{w_j}); [0,1])$,

$$\tilde{\rho_j}\, e_j(x, hD)\left(hD_{x_k} - a_j(x, hD')\right)\rho_j v_h = O_{L^2}(h). \tag{7}$$

In (7) and below, we abuse notation somewhat and sometimes write $\rho_j := \rho_j(x, hD)$ and $\tilde{\rho_j} := \tilde{\rho_j}(x, hD)$ when the context is clear.

Since $e_j(x, hD)$ is h-elliptic on V_{w_j} it follows by a standard h-microlocal parametrix construction applied to (7) that

$$\tilde{\rho_j}\big(hD_{x_k} - a_j(x, hD')\big)\rho_j v_h = O_{L^2}(h). \tag{8}$$

In the following, we will employ a number of convenient spatial cutoff functions that we introduce here.

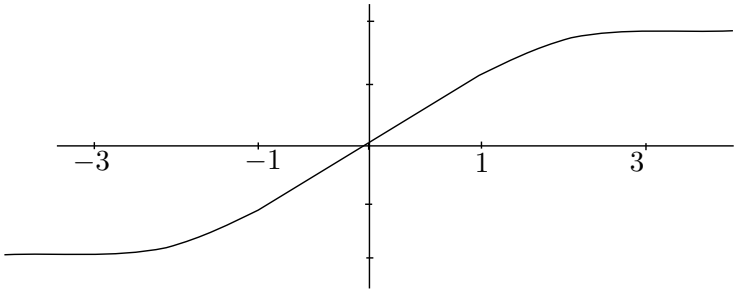

FIGURE 1. A sketch of the function $\tilde{\chi}$ used in the proof of Theorem 1.

Let $\tilde{\chi}(s) \in \mathcal{C}^\infty(\mathbb{R})$ satisfy the following conditions:

- $\tilde{\chi}$ is odd,
- $\tilde{\chi}' \geq 0$,
- $\tilde{\chi}(s) \equiv -1$ for $s \leq -3$ and $\tilde{\chi}(s) \equiv 1$ for $s \geq 3$,
- $\tilde{\chi}(-1) = -1/2$ and $\tilde{\chi}(1) = 1/2$,
- $\tilde{\chi}(s) = \frac{s}{2}$ for $-1 \leq s \leq 1$.

See Figure 1 for a picture. Let $\gamma(s) = \tilde{\chi}'(s)$ so that γ has support in $\{-3 \leq s \leq 3\}$, $\gamma(s) \geq 0$, and $\gamma(s) \equiv 1/2$ for $|s| \leq 1$.

In the following, it will also be useful to define the corresponding rescaled functions

$$\tilde{\chi}_\mu(s) := \tilde{\chi}(\mu^{-1}s), \quad \gamma_\mu(s) := \gamma(\mu^{-1}s), \ \ \mu \geq h.$$

Next, setting $\gamma(x) := \prod_{k=1}^n \gamma(x_k)$, we multiply (8) by $\tilde{\chi}_\mu(x_k) \cdot \gamma(x)$ and integrate against $\rho_j v_h(x)$ to get

$$\begin{aligned} &\langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)\, hD_{x_k}\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \\ &\qquad = \langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)a_j(x,hD')\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \,+\, O(h). \end{aligned} \tag{9}$$

First, we note that since $\tilde{\rho}_j(x,\xi) = 1$ for $(x,\xi) \in \operatorname{supp}\rho_j$, so the symbol of the commutator

$$\sigma([\tilde{\rho}_j, a_j(x,hD')])(x,\xi) \sim 0, \quad (x,\xi) \in \operatorname{supp}\rho_j.$$

Consequently, from (9) it follows that for the first term on the RHS of (9),

$$\begin{aligned} &\langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j a_j(x,hD')\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \\ &\qquad = \langle \tilde{\chi}_\mu(x_k)\gamma(x) a_j(x,hD')\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \,+\, O(h^\infty). \end{aligned} \tag{10}$$

Since $a_j(x,hD')^* = a_j(x,hD') + O(h)_{L^2\to L^2}$, $[a_j(x,hD'), \tilde{\chi}_\mu(x_k)] = 0$ and by L^2-boundedness,

$$[a_j(x,hD'), \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)] = O(h)_{L^2\to L^2},$$

it follows from (10) that

$$\operatorname{Im}\langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)a_j(x,hD')\rho_j v_h,\, \rho_j v_h\rangle_{L^2} = O(h). \tag{11}$$

As for the LHS of (9), we again use the fact that $\tilde{\rho}_j = 1$ on $\mathrm{supp}\rho_j$ and

$$\sigma([\tilde{\rho}_j, hD_{x_k}])(x,\xi) \sim 0, \quad (x,\xi) \in \mathrm{supp}\,\rho_j.$$

to get that

$$\begin{aligned} &\langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)\, hD_{x_k}\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \\ &\qquad = \langle \tilde{\chi}_\mu(x_k)\gamma(x)\, hD_{x_k}\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \,+\, O(h). \end{aligned} \tag{12}$$

Taking imaginary parts of both sides of (12) gives

$$\begin{aligned} &\mathrm{Im}\ \langle \tilde{\chi}_\mu(x_k)\gamma(x)\tilde{\rho}_j(x,hD)\, hD_{x_k}\rho_j v_h,\, \rho_j v_h\rangle_{L^2} \\ &\qquad = \frac{h}{2}\int\cdots\int \tilde{\chi}_\mu(x_k)\gamma(x)\,\partial_{x_k}\,|\rho_j v_h|^2\,dx'dx_k \,+\, O(h). \end{aligned} \tag{13}$$

From (9), (13) and (11) it then follows that

$$\frac{h}{2}\int\cdots\int \tilde{\chi}_\mu(x_k)\gamma(x)\,\partial_{x_k}|\rho_j v_h(x)|^2\,dx_k dx' = O(h). \tag{14}$$

Next, one integrates by parts on the LHS in the x_k-variable (14). The boundary terms vanish since ρ_j has compact support in the x-variables and differentiation of $\gamma(x)$ term contributes an $O(h)$ term that can be absorbed in the RHS of (14). The result is that

$$\frac{h}{\mu}\int\cdots\int \gamma_\mu(x_k)\gamma(x)\,|\rho_j v_h(x)|^2\,dx_k\,dx' = O(h). \tag{15}$$

Since $\gamma_\mu(s) \in C_0^\infty(\mathbb{R};[0,1])$ with $\gamma_\mu(s) \equiv \frac{1}{2}$ for $|s| \le \mu$, it follows from (15) that

$$\frac{h}{\mu}\int_{|x_k|\le\mu}\int_{|x'|\le 1} |\rho_j v_h(x)|^2\,dx_j dx' = O(h); \quad k = 1,2,...,n. \tag{16}$$

Clearly the bound in (16) holds for all indices $j = 1,...N$. Recalling that $\sum_j \rho_j v_h = v_h + O_{C^\infty}(h^\infty)$, we use the pointwise bound $|v_h|^2 \le N\sum_j |\rho_j v_h|^2 + O(h^\infty)$, and cancel the h on both sides of (16) to get

$$\begin{aligned} \mu^{-1}\int_{|x_1|\le\mu}\cdots\int_{|x_n|\le\mu}|v_h(x)|^2\,dx &\le N\mu^{-1}\sum_{j=1}^N \int_{|x_1|\le\mu}\cdots\int_{|x_n|\le\mu}|\rho_j v_h(x)|^2\,dx \\ &\le N\mu^{-1}\sum_{j=1}^N\int_{|x_k|\le\mu}\int_{|x'|\le 1}|\rho_j v_h(x)|^2\,dx = O(1). \end{aligned} \tag{17}$$

Since cubes of width μ are comparable to balls of radius μ, the estimate in (17) proves Theorem 1 in the interior of Ω on *all* scales $0 < \mu \le 1$.

2.2. **Estimates at the boundary on scales $\mu \geq h$.** Let $p_0 \in \partial\Omega$ and choosing Fermi coordinates $(x_n, x') : B_1(0) \to \mathbb{R}^n$ with $(0,0) = p_0$ and $\partial\Omega = \{x_n = 0\}$. Let $B_\mu(0) = \{x; |x| < \mu\}$ be a radius $\mu \geq h$ ball centered at p_0. Let $\psi \in C_0^\infty(B_1(0))$ with $\psi(x', x_n) = 1$ when $|x_n| < 1$ so that $\psi(\mu^{-1}x) = 1$ for $|x_n| \leq \mu$, and set

$$A(h) := \psi(\mu^{-1}x) \cdot \big(x_n h D_{x_n} + x' \cdot hD_{x'}\big), \quad x \in B_\mu(0). \tag{18}$$

Since $g_{ij}(x) = \delta^i_j + O(|x|)$ it follows that

$$\begin{aligned} -h^2\Delta\phi(x) &= [(hD_n)^2 + (hD_{x'})^2]\phi(x) + E_{\mu,h}(x), \quad x \in B_\mu(0), \\ \|E_{\mu,h}\|_{L^2(B_\mu)} &= O(\mu)\|\phi\|_{H^2_h(\Omega)} = O(\mu), \end{aligned} \tag{19}$$

since $\|\phi\|_{H^2_h(\Omega)} = O(1)$ from the elliptic estimates

$$\|\phi\|_{H^2_h(\Omega)} \leq C_1\Big(\|(-h^2\Delta + 1)\phi\|_{L^2(\Omega)} + \|\phi_h\|_{L^2(\Omega)}\Big) \leq 3C_1. \tag{20}$$

Choosing $A(h)$ as in (18), it follows by the Rellich lemma that in *both* the Dirichlet and Neumann cases

$$\frac{i}{h}\Big\langle [-h^2\Delta - 1, A]\phi, \phi\Big\rangle_\Omega = \langle h\partial_n\phi, A\phi\rangle_{\partial\Omega} - \langle \phi, h\partial_n(A\phi)\rangle_{\partial\Omega} = 0. \tag{21}$$

In the Dirchlet case, the last equality in (21) follows since $\partial\Omega = \{x_n = 0\}$ so that $\phi = 0$ and both $x_n hD_n\phi|_{\partial\Omega} = 0$ and $x' \cdot hD_{x'}\phi|_{\partial\Omega} = 0$. As a result, $A\phi|_{\partial\Omega} = 0$. In the Neumann case, $h\partial_n\phi|_{\partial\Omega} = 0$ and $h\partial_n A\phi|_{\partial\Omega} = x' hD_{x'} h\partial_n\phi|_{\partial\Omega} = 0$.

From (19) and (21) it then follows that

$$\begin{aligned} \frac{i}{h}\Big\langle [-h^2\Delta - 1, A]\phi, \phi\Big\rangle_\Omega &= \frac{i}{h}\Big\langle \psi(\mu^{-1}x)[-h^2\Delta - 1, x \cdot hD_x]\phi, \phi\Big\rangle_\Omega \\ &\quad + \frac{i}{h}\Big\langle [-h^2\Delta - 1, \psi(\mu^{-1}x)]\, x \cdot hD_x\phi, \phi\Big\rangle_\Omega = 0. \end{aligned} \tag{22}$$

Since

$$\begin{aligned} &\frac{i}{h}\Big\langle [-h^2\Delta - 1, \psi(\mu^{-1}x)]\, x \cdot hD_x\phi, \phi\Big\rangle_\Omega \\ &\quad = \Big\langle \frac{h}{\mu}\nabla\psi(x/\mu) \cdot hD_x\ (x \cdot hD_x)\phi, \phi\Big\rangle_\Omega + \big(\frac{h}{\mu}\big)^2\Delta\psi(x/\mu)\ (x \cdot hD_x)\,\phi, \phi\Big\rangle_\Omega \\ &\quad \leq \big(C_1(h\mu^{-1})^2\mu + C_2(h\mu^{-1})\mu\big) \cdot \|\phi\|^2_{H^2_h(\Omega)} \leq C_3\mu, \end{aligned} \tag{23}$$

where in the last line of (23) we have used that $\mu \geq h$ along with the elliptic estimates (20).

Thus, from (23) and (22),

$$\frac{i}{h}\Big\langle \psi(\mu^{-1}x)[-h^2\Delta - 1, x \cdot hD_x]\phi, \phi\Big\rangle_\Omega = O(\mu). \tag{24}$$

From (19), it follows that on the support of $\psi(\mu^{-1}x)$,

$$\begin{aligned}\frac{i}{h}[-h^2\Delta, x\cdot hD_x]\phi &= \frac{i}{h}[(hD_n)^2+(hD_{x'})^2,\, x\cdot hD_x]\phi + O(\mu)\|\phi\|_{H^2_h(\Omega)}\\ &= -h^2\Delta\phi + O(\mu)\|\phi\|_{H^2_h(\Omega)}\\ &= 2\phi + O(\mu). \end{aligned} \tag{25}$$

Finally, substitution of (25) in (24) gives $\|\phi\|^2_{B_\mu(0)} = O(\mu)$ when $\mu \geq h$ and completes the proof of Theorem 1.

□

Remark 3. We note that the factorization argument in Section 2.1 is difficult to apply along general C^2 boundaries and the Rellich commutator argument above fails when $\mu < h$ since the bounds on the RHS of (23) blow up in such cases.

3. Non-concentration and L^∞ bounds for eigenfunctions

In the case of C^∞ compact manifolds without boundary, the eigenfunction sup bounds

$$\|\phi_h\|_{L^\infty(\Omega)} = O(h^{\frac{1-n}{2}}) \tag{26}$$

were proved by Avakumovic [Av] and Levitan [L] in the case of Laplace eigenfunctions and the bound was proved for eigenfunctions of general self-adjoint elliptic operators by Hörmander [Ho]. For Dirichlet or Neumann eigenfunctions on C^∞ manifolds with boundary, the analogous bound was proved by Grieser [Gr]. In this section, we use Theorem 1 to give an alternative entirely local and stationary proof of the eigenfunction sup bounds (26) for Dirichlet or Neumann eigenfunctions in the case of manifolds with C^∞ boundary.

First, we note that for any $p \in \Omega$, from the eigenfunction sup bounds $\|\phi_h\|_{L^\infty(\Omega)} = O(h^{\frac{1-n}{2}})$ it is immediate that

$$\int_{B(p_0,h)} |\phi_h|^2 dvol \leq C_n h^n h^{1-n} = C_n h.$$

Thus, on the fundamental wave length scale $\mu = h$, the non-concentration bound in Theorem 1 follows immediately from the eigenfunction sup bounds.

We note that there is actually a converse to this result. In the case where $\partial\Omega = \emptyset$, the result below was proved by Sogge in [So] (see (3.3) on pg. 391) using wave methods. We extend this result to Dirichlet or Neumann eigenfunctions on manifolds with boundary using an entirely local stationary argument combined with the maximum principle.

Theorem 2. *Let (Ω^n, g) be a compact C^∞ Riemannian manifold with or without boundary and ϕ_h be an L^2-normalized Laplace eigenfunction satisfying Dirichlet or Neumann boundary conditions in the case where $\partial\Omega \neq \emptyset$. Then, there exist a uniform constant $C = C(\Omega^n, g)$ such that*

$$\|\phi_h\|_{L^\infty(\Omega)} \leq C h^{-\frac{n}{2}} \sup_{x\in\Omega} \|\phi_h\|_{L^2(B(x,h)\cap\Omega)}.$$

Proof. We first prove this outside an h-width Fermi neignbourhoods of the boundary given by

$$\Omega_{int}(h) := \{x \in \Omega; d(x, \partial\Omega) \geq h\}.$$

3.1. **Sup bounds at interior points.** We first assume here that $p_0 \in \Omega_{int}(h)$ and let $x : B \to \mathbb{R}^n$ be geodesic normal coordinates defined in an open ball $B = B(p_0, \varepsilon_0)$with $x(p_0) = 0$. In the following, we let $\Delta_0 = \sum_j \partial_{x_j}^2$ be the local flat Laplacian in the x-coordinates and note that in the small ball $B_h = \{|x| \leq h\}$, there exists a second-order differential operator $A(x, D_x)$ with

$$\begin{aligned} \Delta\phi &= \Delta_0\phi + A(x, D_x)\phi, \\ \|A(x, D_x)\phi\|_{L^2(B_h)} &= O(h^2)\|\phi\|_{H^2(B_h)} + O(h)\|\phi\|_{H^1(B_h)}, \end{aligned} \tag{27}$$

where the last line in (27) follows from the fact that $g^{ij}(x) = \delta^i_j + O(|x|^2)$ with $\nabla g^{ij}(x) = O(|x|)$.

From standard elliptic estimates

$$\|\phi\|_{H^{s+2}(B_h)} \leq C_1\|(1-\Delta)\phi\|_{H^s(B_h)} + C_2\|\phi\|_{L^2(B_{2h})} \tag{28}$$

and the non-concentration bound $\|\phi\|^2_{B_{2h}} = O(h)$ it then follows from (27) that

$$\|A(x, D_x)\phi\|_{L^2(B_h)} = O(1)\|\phi_h\|_{L^2(B_{2h})} = O(h^{1/2}). \tag{29}$$

Assume first that $n \geq 3$. Given $x_0 \in B$ and $\mu \geq h$ we set $B_\mu(x_0) := \{y; |y - x_0| \leq \mu\}$ and $S_\mu(x_0) := \{y; |y - x_0| = \mu\}$ in the following. Since ϕ is approximately harmonic when $\mu \leq h$, we mimic the proof of the mean value theorem to derive the required a priori estimates. An application of Green's formula with $\phi(y)$ and the local Green's function $G(x_0, y) = c_n|x_0 - y|^{2-n}$ with $(\Delta_0)_y G(x_0, y) = \delta_{x_0}(y)$ gives

$$\begin{aligned} &\phi(x_0) - \int_{B_\mu(x_0)} \Delta_0\phi(y)\ G(x_0, y)\, dy \\ &= c'_n\mu^{1-n}\int_{S_\mu(x_0)} \phi(y)d\sigma(y) - c_n\mu^{2-n}\int_{S_\mu(x_0)} \partial_r\phi(y)d\sigma(y), \end{aligned} \tag{30}$$

where $c'_n = (2-n)c_n$ and $r = |y - x_0|$.

Substitution of (27) and (29) in the LHS of (30) gives

$$\begin{aligned} &\phi(x_0) - c_n h^{-2}\int_{B_\mu(x_0)} \phi(y)|x_0 - y|^{2-n}dy + \int_{B_\mu(x_0)} A\phi(y)|x_0 - y|^{2-n}\, dy \\ &\qquad = c'_n\mu^{1-n}\int_{S_\mu(x_0)} \phi(y)d\sigma(y) - c_n\mu^{2-n}\int_{S_\mu(x_0)} \partial_r\phi(y)d\sigma(y), \end{aligned} \tag{31}$$

By another application of Green and (29),

$$\begin{aligned} -\int_{S_\mu(x_0)} \partial_r\phi(y)d\sigma(y) &= -\int_{B_\mu(x_0)} \Delta\phi(y)dy + \int_{B_\mu(x_0)} A\phi(y)dy \\ &= h^{-2}\int_{B_\mu(x_0)} \phi(y)dy + O(\mu^{\frac{n}{2}}\|\phi_h\|_{L^2(B_\mu)}). \end{aligned} \tag{32}$$

Substitution of (32) in (31) gives

$$\phi(x_0) - c_n h^{-2} \int_{B_\mu(x_0)} \phi(y)|x_0 - y|^{2-n} dy + c_n \int_{B_\mu(x_0)} A\phi(y)|x_0 - y|^{2-n}\, dy$$
$$= c_n' \mu^{1-n} \int_{S_\mu(x_0)} \phi(y) d\sigma(y) + c_n \mu^{2-n} h^{-2} \int_{B_\mu(x_0)} \phi(y) dy + O(\mu^{2-\frac{n}{2}} \|\phi_h\|_{L^2(B_\mu)}). \quad (33)$$

We rewrite the first term on the RHS of (33) as a ball integral using Green yet again with functions $\phi(y)$ and $\rho(y) = |y - x_0|^2$. This gives

$$-h^{-2} \int_{B_\mu} \phi\, \rho dy - 2 \int_{B_\mu} \phi dy = \mu^2 \int_{S_\mu} \partial_r \phi d\sigma(y) - 2\mu \int_{S_\mu} \phi d\sigma(y)$$
$$= -\mu^2 h^{-2} \int_{B_\mu} \phi dy - 2\mu \int_{S_\mu} \phi d\sigma(y) + O(\mu^2 h^{\frac{n}{2}} \|\phi_h\|_{L^2(B_h)}).$$

where in the last line we have input (32).

Setting $\mu = h$ and solving for $h^{1-n} \int_{S_\mu} \phi\, d\sigma$ gives:

$$2h^{1-n} \int_{S_h(x_0)} \phi d\sigma = h^{-2-n} \int_{B_h(x_0)} \phi \cdot \rho dy + 2h^{-n} \int_{B_h(x_0)} \phi dy$$
$$- h^{-n} \int_{B_h(x_0)} \phi dy + O(h^{-\frac{n}{2}} h^2 \|\phi_h\|_{L^2(B_h)}). \quad (34)$$

Finally, substitution of (34) in (33) gives

$$\phi(x_0) = c_n h^{-2} \int_{B_h(x_0)} \phi(y)|x_0 - y|^{2-n} dy - c_n \int_{B_h(x_0)} A\phi(y)|x_0 - y|^{2-n}\, dy$$
$$+ \frac{c_n'}{2} h^{-2-n} \int_{B_h(x_0)} \phi(y) \cdot \rho(y)\, dy - c_n' \frac{h^{-n}}{2} \int_{B_h(x_0)} \phi(y) dy +$$
$$+ c_n' h^{-n} \int_{B_h(x_0)} \phi(y) dy + c_n h^{-n} \int_{B_h(x_0)} \phi(y) dy + O(h^{-\frac{n}{2}} h^2 \|\phi_h\|_{L^2(B_h)}). \quad (35)$$

To estimate the first integral on the RHS, we introduce polar variables with $r(y) = |x_0 - y|$ and integrate by parts $n - 3$ times with respect to ∂_r. More concretely, let $\chi \in C_0^\infty(B_1)$ be a radial cutoff with $\chi = \chi(r)$ satisfying $\chi|_{B_{1/2}} = 1$ and $0 \le \chi \le 1$.

Then, setting $I_1(h) := h^{-2} \int_{B_h(x_0)} \phi(y)|x_0 - y|^{2-n} dy$, we write

$$I_1(h) = h^{-2} \int_{B_h} |y - x_0|^{2-n} \phi_h(y)\, \chi(h^{-1} r) dy + h^{-2} \int_{B_h} |y - x_0|^{2-n} \phi_h(y)\, (1 - \chi)(h^{-1} r) dy \quad (36)$$

To bound the second term on the RHS of (36), we apply Cauchy-Schwarz to get

$$h^{-2}\Big|\int_{B_h}|y-x_0|^{2-n}\phi_h(y)\,(1-\chi)(h^{-1}r)dy\Big| \leq C_n h^{-n}\|\phi_h\|_{L^1(B_h)}$$
$$\leq C_n h^{-n}|B_h|^{1/2}\,\|\phi\|_{L^2(B_h)} \leq C_n h^{-\frac{n}{2}}\|\phi\|_{L^2(B_h)}. \tag{37}$$

As for the first term on the RHS of (36) we note that $f(y)=|y-x_0|^{2-n}\notin L^2_{loc}$ when $n>3$. To remedy this, we pass to polar coordinates with $r=|y-x_0|$ and integrate by parts $n-2$-times with respect to ∂_r. Since boundary term vanishes at $r=0$,

$$h^{-2}\int_{B_h}|y-x_0|^{2-n}\phi_h(y)\,\chi(h^{-1}r)dy = C_n h^{-2}\int_{B_h}\partial_r^{n-2}\big(\phi_h(y)\chi(h^{-1}r)\big)\,dy, \tag{38}$$

and by Cauchy-Schwarz,

$$\Big|C_n h^{-2}\int_{B_h}\partial_r^{n-2}\big(\phi_h(y)\chi(h^{-1}y)\big)\,dy\Big| \leq C_n h^{-2}\Big(\sum_{k=0}^{n-2}h^{-k}\|\phi\|_{H^{n-2-k}(B_h)}\Big)\,|B_h|^{\frac{1}{2}}. \tag{39}$$

From the elliptic estimates (28), $\|\phi\|_{H^{n-2-k}(B_h)} = O(h^{-(n-2-k)})\|\phi\|_{L^2(B_{2h})}$ and so from (39) it follows by another application of Cauchy-Schwarz that

$$\Big|h^{-2}\int_{B_h}|y-x_0|^{2-n}\phi_h(y)\,\chi(h^{-1}y)dy\Big| \leq C_n h^{-2}h^{2-n}\,|B_h|^{\frac{1}{2}}\,\|\phi\|_{L^2(B_{2h})} \tag{40}$$
$$= O(h^{-\frac{n}{2}})\|\phi\|_{L^2(B_{2h})}.$$

Consequently, from (40) and (37), $I_1(h)=O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_{2h})})$.

The second integral is estimated in the same way using integration by parts with respect to ∂_r^{n-2}, Cauchy -Schwarz and the elliptic estimates

$$\|A\phi\|_{H^{n-2-k}(B_h)} = \mathcal{O}(h^2)\|\phi\|_{H^{n-k}(B_h)} = O(h^{-(n-2-k)})\|\phi\|_{L^2(B_{2h})}.$$

The result is that

$$\Big|\int_{B_h}A\phi(y)|x_0-y|^{2-n}dy\Big| \leq C_n\Big(\sum_{k=0}^{n-2}h^{-k}\|A\phi\|_{H^{n-2-k}(B_h)}\Big)\,|B_h|^{\frac{1}{2}}$$
$$\leq C_n h^{2-n}|B_h|^{\frac{1}{2}}\|\phi\|_{L^2(B_{2h})} = O(h^{-\frac{n}{2}}h^2\|\phi\|_{L^2(B_{2h})}). \tag{41}$$

The rest of the terms on the RHS of (35) are all estimated in a similar fashion. By Cauchy-Schwarz and Theorem 1,

$$h^{-n}\Big|\int_{B_h(x_0)}\phi(y)dy\Big| \leq C_h h^{-n}\,|B_h(x_0)|^{1/2}\,\|\phi\|_{L^2(B_h)}$$
$$\leq C_n h^{-n}h^{\frac{n}{2}}\|\phi\|_{L^2(B_{2h})}.$$

In addtion, since $\max_{y\in B_h}\rho(y)\leq h^2$,

$$h^{-2-n}\int_{B_h(x_0)}\phi\cdot\rho\,dy = O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_{2h})}).$$

The upshot is that $|\phi(x_0)| = O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_{2h})})$ for any $x_0 \in \Omega_{Int}(h)$ and since this bound is uniform, it follows that

$$\sup_{x\in\Omega_{Int}(h)} |\phi(x)| \leq C_n h^{-\frac{n}{2}} \sup_{x\in\Omega_{Int}(h)} \|\phi\|_{L^2(B_h(x))}. \tag{42}$$

3.2. **Sup bounds in the boundary layer $\partial\Omega(h) = \Omega \setminus \Omega_{int}(h)$.** Here, we consider points $p_0 \in \partial\Omega(h) = \{x \in \Omega, d(x, \partial\Omega) \leq h\}$. To derive the requisite eigenfunction boundary at such points, let $\tilde{\Omega} \ni \overline{\Omega}$ be an open n-manifold extension of $\overline{\Omega} = \Omega \cup \partial\Omega$ and make a C^∞-extension of the Riemannian metric on $\overline{\Omega}$ to $\tilde{\Omega}$. We abuse notation somewhat and denote the extension by g also and refer to $U := \tilde{\Omega} \setminus \overline{\Omega}$ as the *collar extension* of Ω. We let $(y', y_n) : U \to \mathbb{R}^n$ denote Fermi coordinates in U with $\partial\Omega = \{y_n = 0\}$ and let ϕ_h denote an even (resp. odd) extension to U of the Neumann (resp. Dirichlet) eigenfunction ϕ_h in Ω. By regularity of eigenfunctions up to the boundary, $\phi \in C^\infty(\overline{\Omega}) \cap C^\infty(\overline{\Omega^c})$ and so, It follows by Taylor expansion in y_n that the extension $\phi_h \in C^2(\tilde{\Omega})$ in both the Neumann and Dirichlet cases. Given $p_0 \in \partial\Omega$ we let $B \subset \tilde{\Omega}$ be a geodesic ball in the extended metric centered at p_0 and $x : B \to \mathbb{R}^n$ be geodesic normal coordinates with $x(p_0) = 0$ and let $B^\pm = \{\pm y_n \geq 0\} \cap B$ be upper (resp. lower) closed half-balls contained in Ω (resp. Ω^c). Since $g^{ij}(x) = \delta^i_j + O(|x|^2)$ and $\nabla g^{ij}(x) = O(|x|)$ it follows from elliptic estimates as in (27) and (29) (recall that $\phi \in C^2(B)$) and the fact that ϕ and Δ_0 are invariant under the involution $y_n \to -y_n$ that $\Delta_0\phi = -h^{-2}\phi + A_\pm(x, D)\phi$, where $A_\pm(x, D_x)$ are second order differential operators with $A_-(x, D_x) = A_+(-x, -D_x)$, it follows by elliptic regularity that

$$\Delta_0\phi = -h^{-2}\phi + A_\pm(x, D)\phi, \quad \|A_\pm\phi\|_{L^2(B_h^\pm)} = O(1)\|\phi\|_{L^2(B_h^\pm)} = O(h^{\frac{1}{2}}), \tag{43}$$

where $B_h = \{|x| \leq h\} \subset B$ and the last equality in (43) follows from Theorem 1. Thus, even though the extension ϕ is *not* in general an eigenfunction of the extended Laplacian on $\tilde{\Omega}$, it is nevertheless a reasonably good quasimode for the flat Laplacian $\Delta_0 = \sum_j \partial^2_{x_j}$ on small balls B_h centered on points of the boundary. For convenience, in the following we denote the reflection map $r : B^\pm \to B^\mp$ by $r(y', y_n) = (y', -y_n) =: y^*$.

Remark 4. We note that in the above the metric extensions to $\tilde{\Omega}$ do *not* involve reflection in the boundary to avoid the usual issues with only Lipschtiz regularity of the metric. Indeed, we simply extend g in a smooth fashion across the boundary $\partial\Omega$ and then reflect the eigenfunctions in the adapted Fermi coordinates (y', y_n).

3.2.1. *Estimates at boundary points.* We first estimate eigenfunctions at boundary points, $p_0 \in \partial\Omega$. To estimate $|\phi(p_0)|$ we first note making the change of variables $y_n \to -y_n$, it follows that in both Neumann and Dirichlet cases

$$\frac{1}{2}\|\phi\|_{L^2(B_h)} = \|\phi\|_{L^2(B_h^-)} = \|\phi\|_{L^2(B_h^+)} = O(h^{\frac{1}{2}}),$$

where the last estimate follows from Theorem 1. Since $\phi \in C^2(B_n)$, one can apply the Green's formula argument just as in the interior case in subsection 3.1 to get the formula for $\phi(p_0)$ in (35). Splitting the integrals on the RHS into pieces over the half-balls $B_h^\pm$ and then making the change of variable $B_h^- \ni y \mapsto y^* \in B^+$ and using

that $\phi(y^*) = \pm\phi(y)$, (depending on whether boundary conditions are Neumann or Dirichlet), one can write

$$\begin{aligned}\phi(p_0) =& c_n h^{-2} \int_{B_h^+(p_0)} \phi(y)\big(|p_0-y|^{2-n} \pm |p_0-y^*|^{2-n}\big)dy \qquad (44)\\ &- \int_{B_h^+(p_0)} A_+\phi(y)|p_0-y|^{2-n}\,dy\\ &- \int_{B_h^+(p_0)} A_-\phi(y)|p_0-y^*|^{2-n}\,dy\\ &+ \frac{c_n'}{2}h^{-2-n}\int_{B_h^+(p_0)} \phi(y)\cdot\big(\rho(y)+\rho(y^*)\big)\,dy\\ &- 2c_n'\frac{h^{-n}}{2}\int_{B_h^+(p_0)}\phi(y)dy + 2c_n'h^{-n}\int_{B_h^+(p_0)}\phi(y)dy\\ &+ 2c_n h^{-n}\int_{B_h^+(p_0)}\phi(y)dy + O(h^{-\frac{n}{2}}h^2\|\phi\|_{L^2(B_h^+)}). \qquad (45)\end{aligned}$$

One estimates all terms on the RHS of (44) just as in the interior case. Indeed, since $\phi(y^*) = \pm\phi(y), |y^*-p_0| = |y-p_0|$ and $dy = dy^*$, the first term is just $\int_{B_h^+(p_0)}\phi(y)|p_0-y|^{2-n}dy \pm \int_{B_h^+(p_0)}\phi(y)|p_0-y|^{2-n}\big)dy$ where $+$(resp. $-$) corresponds to Neumann (resp. Dirichlet) boundary conditions. Putting in the radial cutoff $\chi(h^{-1}r)$ as in the previous section, it follows by Cauchy-Schwarz that

$$h^{-2}\int_{B_h^+(p_0)}\phi(y)(1-\chi(h^{-1}r))|p_0-y|^{2-n}dy = O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_h^+)}).$$

To estimate the first term on the RHS, the key point here is that by C^∞ regularity of the eigenfunctions up to the boundary, $\phi \in C^\infty(B_h^+(p_0))$ and since $\partial_r(p) \in T_p(\partial\Omega)$ for all $p \in \partial\Omega$, the integration by parts argument in (38) with respect to the radial variable $r = |y-p_0|$ can be used to estimate the first term in (44) just as in the interior case. In particular, just as in (38), (39) and (40),

$$\begin{aligned}h^{-2}\int_{B_h^+(p_0)}|y-x_0|^{2-n}\phi_h(y)\,\chi(h^{-1}r)dy = C_n h^{-2}\int_{B_h^+(p_0)}\partial_r^{n-2}\big(\phi_h(y)\chi(h^{-1}r)\big)\,dy\\ = O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_h^+)}). \qquad (46)\end{aligned}$$

A similar argument applies to the terms $\int_{B_h^+(p_0)} A_\pm\phi(y)|p_0-y|^{2-n}\,dy$ and it follows as in (41) that

$$\int_{B_h^+(p_0)} A_\pm\phi(y)|p_0-y^*|^{2-n}\,dy = O(h^{-\frac{n}{2}}h^2\|\phi\|_{L^2(B_h^+)}).$$

Observing that $\Delta_0(\rho(y)+\rho(y^*))=4$, the remaining terms on the RHS of (44) are all $O(h^{-\frac{n}{2}}\|\phi\|_{L^2(B_h^+)})$ by straightforward applications of Cauchy-Schwarz just as in the interior case. Since the bounds for $|\phi(p_0)|$ are uniform for $p_0\in\partial\Omega$ It follows that

$$\max_{p\in\partial\Omega}|\phi(p)|\leq C_n h^{-\frac{n}{2}}\max_{p\in\partial\Omega}\|\phi\|_{L^2(B_h^+(p))} \tag{47}$$

We note that since in the Dirichlet case $\phi|_{\partial\Omega}=0$, the bound in (47) is only relevant in the Neumann setting.

3.2.2. *Estimates in the boundary layer.* Consider the boudary layer $\partial\Omega(8^{-1}\pi h)=\{0\leq x_n\leq\frac{\pi h}{8}\}$ where $(x',x_n):\Omega(\epsilon)\to\mathbb{R}^n$ continue to denote Fermi coordinates where $\epsilon>0$ is sufficiently small. Then, $\partial\left(\partial\Omega(\frac{\pi h}{8})\right)=\{x_n=8^{-1}\pi h\}\cup\{x_n=0\}$ where $\partial\Omega=\{x_n=0\}$. To estimate $\|\phi_h\|_{L^\infty(\partial\Omega(\frac{\pi h}{8}))}$, we follow the argument in [Gr] (see Lemma 6) and apply an adapted maximum principle argument. Set

$$\psi_h(x)=\cos(2h^{-1}x_n)\phi_h(x),\quad x\in\partial\Omega(8^{-1}\pi h).$$

A straightforward computation with the Laplacian in Fermi coordinates shows that

$$P\psi_h(x)=0,\quad x\in\partial\Omega(8^{-1}\pi h),$$

where $P=-\sum_{i,j}g^{ij}(x)\partial_i\partial_j+\sum_i a_i(x)\partial_i+c$ where $c>0$.

Then, by the weak maximum principle applied to P and ψ, it follows that

$$\|\psi_h\|_{L^\infty(\partial\Omega(8^{-1}\pi h))}=\max_{\{x_n=8^{-1}\pi h\}}|\psi_h(x)|+\max_{\{x_n=0\}}|\psi_h(x)|. \tag{48}$$

Since $|\psi_h(x)|=|\cos(2x_nh^{-1})|\cdot|\phi_h(x)|\leq|\phi_h(x)|$ it follows from the interior sup bounds (42) that

$$\max_{\{x_n=8^{-1}\pi h\}}|\psi_h(x)|=O(h^{-\frac{n}{2}}\sup_{\{x_n=8^{-1}\pi h\}}\|\phi\|_{L^2(B_h(x))})$$

and from the boundary estimate (47)

$$\max_{\{x_n=0\}}|\psi_h(x)|=O(h^{-\frac{n}{2}}\max_{x\in\partial\Omega}\|\phi\|_{L^2(B_h^+(x))})$$

The boundary layer estimate

$$\|\phi_h\|_{L^\infty(\partial\Omega(8^{-1}\pi h))}=O(h^{-\frac{n}{2}}\sup_{x\in\partial\Omega(8^{-1}\pi h))}\|\phi\|_{L^2(B_h^+(x))} \tag{49}$$

then follows from (48), since $\cos(2x_n)\geq\cos(\pi/4)=2^{-1/2}$ when $|x_n|\leq 8^{-1}\pi$. This completes the proof of Theorem 2. □

Remark 5. (a) As an immediate corollary of Theorems 1 and 2 one obtains the sup bound $\|\phi_h\|_{L^\infty(\Omega)}=O(h^{\frac{1-n}{2}})$ for Dirichlet or Neumann eigenfunctions when Ω is a compact manifold with boundary. We note that unlike the argument in [Gr], the proof here of the sharp eigenfunction sup bounds is entirely stationary and local using only the non-concentration bound up to the boundary in Theorem 1 and the local integral estimate in Theorem 2.

(b) In view of Theorems 1 and 2 that any improvement in the non-concentration bound for $\|\phi_h\|_{L^2(B_h)(x)}$ for all $x \in \Omega$ as in [Han, HR] automatically implies the corresponding improvement in the upper bound for $\|\phi_h\|_{L^\infty(\Omega)}$.

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