CONTROL ESTIMATES FOR 0TH ORDER PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We introduce the control conditions for 0th order pseudodifferential operators $\mathbf{P}$ whose real parts satisfy the Morse–Smale dynamical condition. We obtain microlocal control estimates under the control conditions. As a result, we show that there are no singular profiles in the solution to the evolution equation $(i\partial_t - \mathbf{P})u = f$ when $\mathbf{P}$ has a damping term that satisfies the control condition and $f \in C^\infty$. This is motivated by the study of a microlocal model for the damped internal waves.

1. INTRODUCTION

Zeroth order pseudodifferential operators are used as mathematical models for internal waves since the work of Colin de Verdière–Saint-Raymond [CdVSR20]. When the operator is *self-adjoint* and the Hamiltonian flow of its principal symbol satisfies the Morse-Smale condition, [CdVSR20] proved that the solution to the evolution equation has singular concentration on the attractive Lagrangians for the Hamiltonian flow. Such concentration provides a microlocal interpretation of the formation of attractors in internal waves – see [MBSL97], [Bro16] for the physics literature. Using tools from scattering theory, in particular the radial estimates introduced by Melrose [Mel95], Dyatlov–Zworski [DZ19b] provided an alternative proof for the singular concentration of the solutions with relaxed conditions on the operators. Much work has been done on 0th order operators since then: Colin de Verdière [CdV20] studied 0th order operators in high dimensions and with weaker dynamical assumptions; Wang [Wan19] studied the scattering matrix for 0th order operators, which is analogous to the scattering matrix for the Schrödinger operators in the Euclidean spaces; Galkowski–Zworski [GZ22] studied the resonances for 0th order operators via viscosity limits; Wang [Wan22] studied the dynamics of resonances for 0th order operators, in particular, a Fermi golden rule for embedded eigenvalues is proved. Tao [Tao19] studied the spectral theory for 0th order operators on the circle and found embedded eigenvalues for such operators. Almonacid–Nigam [AN22] obtained numerical results for both the evolution equations and viscosity limits of eigenvalues. The recent work of Dyatlov–Wang–Zworski [DWZ21] proved the formation of internal wave attractors in two-dimensional domains. We also mention that there are other motivation to study 0th order operators: for example, Ralston [Ral73] considered the Poincaré problem and

investigated the spectrum and the eigenvalues of a 0th order operator, in the study of rotating fluids,

Motivated by studying the damping of internal wave attractors [Bec19, BBSM18, BM16], we consider a microlocal model for the damped internal waves in this paper. More precisely, for a closed smooth surface M, we consider *non-self-adjoint* 0th order pseudodifferential operators $\mathbf{P} = P - i\chi \in \Psi^0(M)$, where $P \in \Psi^0(M)$ is self-adjoint and $\chi \in C^\infty(M)$, $\chi \geq 0$. We assume P satisfies the Morse–Smale condition as in [CdVSR20, DZ19b], hence it models the internal waves without damping. The smooth function χ, which is also in $\Psi^0(M)$ as a multiplication operator, is now considered as the "damping" term. We are interested in the evolution problem for $\mathbf{P}$. In particular, we show that if χ satisfies certain "control condition", then the internal wave attractors disappear and the solution s stay bounded in L^2: see Theorem 2.

As indicated in the previous paragraph, the main tools we use to study the damping problem are resolvent estimates, in the form of microlocal control estimates for 0th order operators: see Theorem 1. Since the singularity of solutions to $(\mathbf{P} - \omega)u = 0$ propagates along the bicharacteristics of $\operatorname{Re}\sigma(\mathbf{P})$ (where the direction of propagation is determined by the sign of $\operatorname{Im}\sigma(\mathbf{P})$), one can control the solution from a subset K of the contangent bundle $T^*M \setminus 0$, assuming $T^*M \setminus 0$ can be covered by finitely many images of the subset K propagated in finite time. When the Hamiltonian flow of $\operatorname{Re}\sigma(\mathbf{P})$ satisfies the Morse–Smale condition, and we assume the subset K is small and located on the attractive (or repulsive) limit cycles, the repulsive (or attractive, respectively) limit cycles cannot be covered in the above-mentioned way. In these cases, we will use radial estimates from scattering theory [Mel95] – see [HMV04, VD13, DD13, Dya12, HV18, DZ16] for some of the recent developments. We also mention that since the Morse–Smale condition is only stable under small perturbations, we can only obtain the microlocal control for a small range of the spectral parameters. We refer to [Lio88, Mil05, BZ04] for introductions to control theory and resolvent estimates.

Resolvent estimates often imply stability results for damped equations, especially the damped wave equation. Classical results [RT74, BLR92, BG97] in the damped wave equation, together with recent developments [BJ16, Wan20, BG20, KK22, Kle22b] in the settings of non-compact manifolds, singular, anisotropic, and time-dependent damping, implies that there is exponential energy decay without loss of derivatives if and only if the *geometric control condition* is satisfied: the geometric control condition is a strong dynamical assumption that every geodesic enters the damped region in finite time. Moreover, the solution will be sufficiently smooth if both the damping and the initial data are sufficiently smooth. Even if there are, for example, periodic geodesics or other attracting or repulsing invariant sets which are outside the control regions, solutions are still smooth near the attractors, as in the vast control theory and damped wave literature [CdVP94, Chr07, Chr10, ALN14, CSVW14, BC15, LL17,

Wun17, Sta17, Kle19, DK20, Wan21a, Wan21b, Kle22a, Sun22, KW22], inexhaustively listed here.

In stark contrast, in this paper we study solutions to evolution equations of zeroth order, for which solutions near attractors exhibit some blow-up. The goal of this work is to show if damping is introduced near the attractors, it stabilizes the solutions. Note that in this case, the control conditions we impose are weaker than the geometric control condition.

This problem is new and requires new techniques. For the usual damped wave problem, the coercive nature of $-\Delta$ gives rise to a compact resolvent. Using the analytic Fredholm theory, this implies the damped wave resolvent has a meromorphic extension to a neighbourhood of the real axis. In particular, the poles of the damped wave resolvent are of finite order and discrete, hence a contour deformation in the inverse Fourier transform allows one to conclude the exponential decay of energy.

On the other hand, for the zeroth order pseudodifferential operators studied in this paper, the resolvent is not compact in the L^2 space, so the analytic Fredholm theory does not apply. In particular, a limiting absorption principle is non-trivial. In fact, as one can see in the second remark after Theorem 2, the unique continuation principle can fail for zeroth order operators even with a spatial damping term.

1.1. **Main results.** Let M be a compact surface without boundary. Let $\mathbf{P} := P + iQ \in \Psi^0(M)$ satisfy conditions in §1.2. Let $p, q \in S^0(T^*M \setminus 0; \mathbb{R})$ be the principal symbols of P, Q, where p, q are homogeneous of order 0. Let $\Lambda_\pm$ be the attractive $(+)$/repulsive $(-)$ Lagrangian submanifolds of the Hamiltonian flow of p – see §1.2 for details. We first introduce the *control conditions*:

Definition 1.1. *For a conic set $K \subset T^*M \setminus 0$, we say that K satisfies the forward (backward) control condition (CC$\pm$, $+$ for forward, $-$ for backward), if there exists a conic open subset $\Omega \subset T^*M \setminus 0$, such that $\Omega \subset K$ and every connected component of Λ_+ (or Λ_- respectively) intersects Ω.*

See Figures 1(a) and 1(b) below for illustration of the control conditions. If K satisfies the control condition (CC$\pm$), then in the corresponding spaces specified later, for ω close to 0, the solution to the equation

$$(\mathbf{P} - \omega)u = f \in C^\infty(M)$$

can be microlocally controlled by its own part on K. More precisely, we have

Theorem 1. *Suppose $\mathbf{P} \in \Psi^0(M)$ satisfies the conditions in §1.2 and $q_{-1}, H_p, \Lambda_\pm(\omega)$ are as in §1.2. Let (CC$\pm$) be as in Definition 1.1. Then the following are true:*
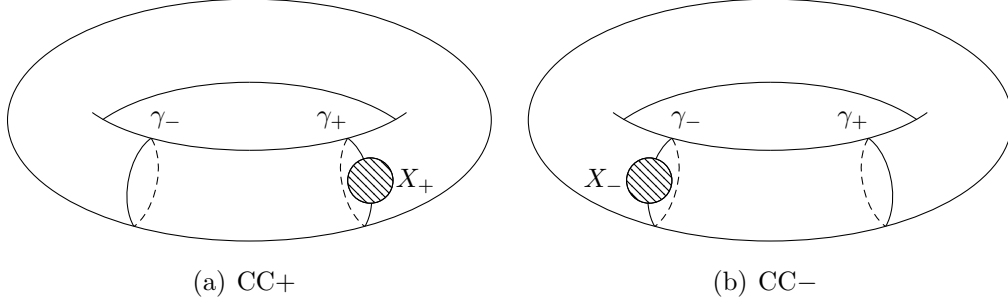
(a) CC+   (b) CC−

FIGURE 1. Illustration of the control conditions. γ_+ is the attractive limit cycle and γ_- is the repulsive limit cycle, see §1.2. The labels $X_\pm$ represents the radial compactification of the elliptic sets of $X_\pm$.

1. *Suppose $X_+ \in \Psi^0(M)$ and $\mathrm{ell}(X_+)$ satisfies (CC+). Then there exists $\delta > 0$ such that for any*

$$|\operatorname{Re}\omega| \leq \delta, \ \operatorname{Im}\omega \geq 0, \ s > \sup_{\Lambda_-(\omega)} \left(-\frac{|\xi|q_{-1}}{H_p|\xi|} \right) - \tfrac{1}{2}, \ N \in \mathbb{R}, \tag{1.1}$$

   there exists $C > 0$ such that for any $u \in C^\infty(M)$,

$$\|u\|_{H^s} \leq C\|X_+u\|_{H^s} + C\|(\mathbf{P} - \omega)u\|_{H^{s+1}} + C\|u\|_{H^{-N}}. \tag{1.2}$$

2. *Suppose $X_- \in \Psi^0(M)$ and $\mathrm{ell}(X_-)$ satisfies (CC-). Then there exists $\delta > 0$ such that for any*

$$|\operatorname{Re}\omega| \leq \delta, \ \operatorname{Im}\omega \geq 0, \ s < \inf_{\Lambda_+(\omega)} \left(-\frac{|\xi|q_{-1}}{H_p|\xi|} \right) - \tfrac{1}{2}, \ N \in \mathbb{R}, \tag{1.3}$$

   there exists $C > 0$ such that for any $u \in C^\infty(M)$,

$$\|u\|_{H^s} \leq C\|X_-u\|_{H^s} + C\|(\mathbf{P} - \omega)u\|_{H^{s+1}} + C\|u\|_{H^{-N}}. \tag{1.4}$$

Remark. The regularity thresholds in (1.1), (1.3) arise naturally from the radial estimates, see [DZ19a, Theorem E.52, Theorem E.54] for instance.

As an application of Theorem 1, we study a microlocal model for the damped internal wave and show the elimination of the singular profile. More precisely, we put

$$\mathbf{P} = P - i\chi, \ \chi \in C^\infty(M; \mathbb{R}_{\geq 0}), \tag{1.5}$$

and we are interested in the long time behavior of the solution to the equation

$$(i\partial_t - \mathbf{P})u(t,x) = f(x), \ u(0,x) = 0. \tag{1.6}$$

If $\chi \equiv 0$ and $0 \notin \operatorname{Spec}_{\mathrm{pp}}(P)$, then by the main theorem of [DZ19b]

$$u(t) = u_\infty + b(t) + \epsilon(t),$$

$$u_\infty \in I^0(M; \Lambda_+(0)), \ \|b(t)\|_{L^2} \leq C, \ \|\epsilon(t)\|_{H^{-1/2-}} \to 0.$$

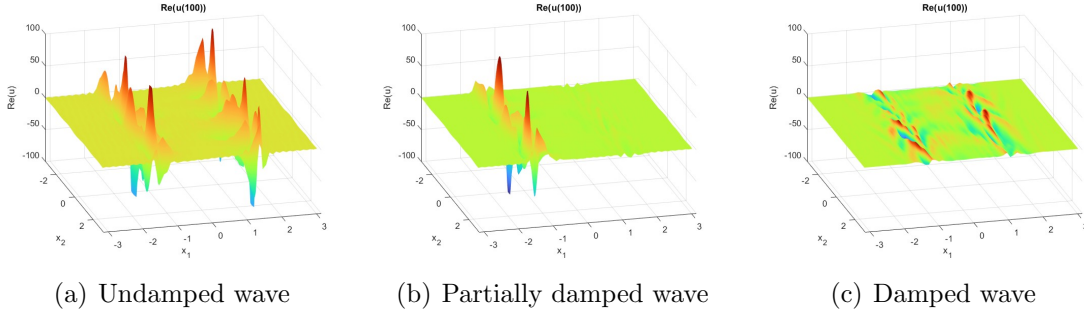(a) Undamped wave   (b) Partially damped wave   (c) Damped wave

FIGURE 2. Numerical illustration of solutions to (1.6) with P, f given in §1.3 and different χ. (a) $\chi_0 = 0$. The undamped wave concentrates at the limit cycles and the L^2 norm of the solution blows up. (b) $\chi_1(x) = \frac{1}{2}e^{-5(x_1-\pi/2)^2}$. χ_1 is numerically localized near $x_1 = \frac{\pi}{2}$ and negligible at $x_1 = -\frac{\pi}{2}$. As a result, the singular profile at $x_1 = \frac{\pi}{2}$ disappears but there is still a singular profile at $x_1 = -\frac{\pi}{2}$. (c) $\chi_2(x_1, x_2) = \chi_1(x_1, x_2) + \chi_1(x_1 + \pi, x_2)$. The damping is localized on both limit cycles and the solution stays bounded in L^2.

Thus the solution has a singular profile u_∞, concentrating on the attracting Lagrangian submanifolds. Figure 2(a) exhibits the concentration.

The following theorem shows that if $0 \notin \mathrm{Spec}_{\mathrm{pp}}(\mathbf{P})$ and χ is nonzero on the projection of every connected component of the attracting Lagrangian to the base manifold, then the solution to (1.6) does not have singular profile: see Figure 2(c).

Theorem 2. *Let $\mathbf{P}$ be as in (1.5) and satisfy (1.8), (1.9), (3.2), and $0 \notin \mathrm{Spec}_{\mathrm{pp}}(\mathbf{P})$. Suppose u solves (1.6), and $T^*(\mathrm{supp}\,\chi) \setminus 0$ satisfies (CC+). Then there exists $C > 0$ that does not depend on t such that*

$$\|u(t)\|_{L^2(M)} \le C, \; t > 0.$$

Remarks. 1. The assumption $0 \notin \mathrm{Spec}_{\mathrm{pp}}(\mathbf{P})$ is equivalent (see the proof of Lemma 3.1) to the following form of the *unique continuation principle* for $P = \mathrm{Re}(\mathbf{P})$:

$$Pu = 0, \; u|_{\mathrm{supp}\,\chi} = 0 \; \Rightarrow \; u \equiv 0. \tag{1.7}$$

In the case where $M = \mathbb{T}^2 := \mathbb{R}^2/(2\pi\mathbb{Z}^2)$ and the full symbol of P admits bounded analytic continuation $p(z, \zeta)$ from $T^*\mathbb{T}^2$ to

$$\left\{ (z, \zeta) \in \mathbb{C}^2/(2\pi\mathbb{Z}^2) |\, |\mathrm{Im}\, z| \le a_1, \; |\mathrm{Im}\, \zeta| \le a_2 \langle \mathrm{Re}\, \zeta \rangle \right\},$$

[Wan22, Proposition 3.1] proved the analyticity of the eigenfunctions u of P. Hence (1.7) is satisfied in this case.

2. For a general $\mathbf{P}$, (1.7) can fail. In fact, let $\mathbf{P} = P - i\chi \in \Psi^0(M)$, $\chi \in C^\infty(M)$ such that $\mathbf{P}$ satisfies conditions in §1.2 and $M \setminus \mathrm{supp}\,\chi \ne \emptyset$. Then for any $v \in C^\infty(M; \mathbb{R})$,

$\operatorname{supp}\chi \cap \operatorname{supp}v = \emptyset$, let Π_{Pv} be the orthogonal projection on to $\mathbb{C}Pv$. Put

$$\widetilde{\mathbf{P}} := (P - \Pi_{Pv}P) - i\chi,$$

Since $\Pi_{Pv}P \in \Psi^{-\infty}(M)$, we know $\widetilde{\mathbf{P}}$ satisfies conditions in §1.2 yet $\widetilde{\mathbf{P}}v = 0$.

3. One can also consider the case where some of the attracting Lagrangians are damped while the others are not. In this case, if Γ_+ is a connected component of $\Lambda_+(0)$ and $T^*(\operatorname{supp}\chi) \setminus 0$ contains a conic open subset that intersects Γ, then u is bounded near Γ_+. If $\Gamma_+ \cap T^*(\operatorname{supp}\chi) \setminus 0 = \emptyset$ instead, then u can have singular profiles near Γ_+ – see Figure 2(b).

1.2. Assumptions.

We assume M is a closed surface, $\mathbf{P} = P + iQ \in \Psi^0(M)$, and P, Q are self-adjoint with respect to some density dm on M:

$$P, Q \in \Psi^0(M), \ P^* = P, \ Q^* = Q \text{ on } L^2(M, dm).$$

P, Q are called the real and the imaginary parts of $\mathbf{P}$ respectively, and we write

$$\operatorname{Re}\mathbf{P} := \frac{\mathbf{P} + \mathbf{P}^*}{2} = P, \ \operatorname{Im}\mathbf{P} := \frac{\mathbf{P} - \mathbf{P}^*}{2i} = Q.$$

Let β be defined as

$$\beta : \mathbb{R}_+ \times (T^*M \setminus 0) \to T^*M \setminus 0, \ \beta(r, x, \xi) := (x, r\xi).$$

β gives an action of $\mathbb{R}_+$ on $T^*M \setminus 0$. We denote the quotient space and the quotient map by

$$\partial\overline{T^*M} := (T^*M \setminus 0)/\mathbb{R}_+, \ \kappa : T^*M \setminus 0 \to \partial\overline{T^*M}.$$

$\partial\overline{T^*M}$ is a 2-dimensional orientable smooth surface.

Let $p, q \in S^0(T^*M \setminus 0; \mathbb{R})$ be the principal symbols of P, Q. Then p, q are homogeneous of order 0. We assume that

$$0 \text{ is a regular value of } p. \tag{1.8}$$

In other words, $dp|_{p^{-1}(0)} \neq 0$. This implies that the characteristic variety $p^{-1}(0)$ is a smooth conic submanifold of $T^*M \setminus 0$. Let H_p be the Hamiltonian vector field

$$H_p := \langle \partial_\xi p, \partial_x \rangle - \langle \partial_x p, \partial_\xi \rangle.$$

Then $|\xi|H_p$ is a smooth vector field on $p^{-1}(0)$ that commutes with β and is homogeneous of order 0. We now define

$$\mathcal{S} := \kappa(p^{-1}(0)), \ \mathrm{X} := \kappa_*(|\xi|H_p).$$

X is a smooth vector field on $\mathcal{S}$. We assume that

$$\varphi_t := e^{t\mathrm{X}} \text{ is a Morse-Smale flow with no fixed points on } \mathcal{S}. \tag{1.9}$$

Recall that (1.9) means that (see for instance [NZ99, Definition 5.1.1])

(1) φ_t has finitely many closed hyperbolic limit cycles;

(2) every trajectory of φ_t that is not a limit cycle has unique limit cycles as its α, ω-limit sets.

We denote the repulsive and attractive limit cycles by γ_-, γ_+ respectively and $\Lambda_\pm := \kappa^{-1}(\gamma_\pm)$. Then $\Lambda_\pm$ are conic Lagrangian submanifolds of $T^*M \setminus 0$. [DZ19b, Lemma 2.1] showed that $\gamma_\pm$ are the radial source $(-)$ and the radial sink $(+)$ respectively, for the Hamiltonian flow of $|\xi|p$, in the sense of [DZ19a, Definition E.50].

Remark. Since regular values and the Morse-Smale property are stable under small perturbations to the spectral parameter, if (1.8), (1.9) are satisfied, then there exists $\delta > 0$ such that for every $\omega \in [-\delta, \delta]$, ω is a regular value of p and (1.9) is satisfied by

$$\mathcal{S}(\omega) := \kappa(p^{-1}(\omega)), \ \mathrm{X}(\omega) := \kappa_*(|\xi|H_p).$$

Moreover, let $\gamma_\pm(\omega)$ be the limit cycles, $\Lambda_\pm(\omega) := \kappa^{-1}(\gamma_\pm(\omega))$, then $\gamma_\pm(\omega)$ are the radial source $(-)$ and the radial sink $(+)$ for the Hamiltonian flow of $|\xi|(p - \omega)$. Moreover, if $K \subset T^*M \setminus 0$ satisfies (CC$\pm$) for $\mathbf{P}$, then it also satisfies (CC$\pm$) for $\mathbf{P} - \omega$, $\omega \in [-\delta, \delta]$. For $\omega \in [-\delta, \delta]$, $\Lambda_\pm(\omega)$ is understood as $\Lambda_\pm(\mathrm{Re}\,\omega)$.

Finally, we assume

$$q \leq 0 \text{ on } T^*M \setminus 0. \tag{1.10}$$

In view of propagation of singularities [DZ19a, Theorem E.47], (1.10) implies that the singularities propagate backwards along the Hamiltonian flow. Let $\alpha \in C^\infty(T^*M; [0,1])$ such that $\mathrm{supp}\,\alpha \subset \{|\xi| \geq 1\}$, $\mathrm{supp}(1-\alpha) \subset \{|\xi| \leq 2\}$. Let q_{-1} be the principal symbol of $Q - \mathrm{Op}(\alpha q) \in \Psi^{-1}(M)$:

$$q_{-1} := \sigma(Q - \mathrm{Op}(\alpha q)).$$

Then q_{-1} is homogeneous of order -1.

1.3. **Examples.** 1. Consider $M = \mathbb{T}^2$,

$$\mathbf{P} = P - i\chi \in \Psi^0(\mathbb{T}^2), \ P := \langle D \rangle^{-1} D_{x_2} - a \cos x_1,$$

$$a > 0, \ a \neq 1, \ \chi \in C^\infty(\mathbb{T}^2), \ \chi \geq 0.$$

P has the principal symbol $p(x, \xi) = |\xi|^{-1}\xi_2 - a \cos x_1$. The radially compactified characteristic variety $\mathcal{S}$ is a disjoint union of two tori. If $0 < a < 1$, then $\mathcal{S}$ covers $\mathbb{T}^2$; if $a > 1$, then $\mathcal{S}$ does not cover $\mathbb{T}^2$. On $p^{-1}(0) \subset T^*\mathbb{T}^2 \setminus 0$, we have

$$H_{|\xi|p} = |\xi|H_p = -\frac{\xi_1\xi_2}{|\xi|^2}\partial_{x_1} + \frac{\xi_1^2}{|\xi|^2}\partial_{x_2} - a(\sin x_1)|\xi|\partial_{\xi_1}.$$

$H_{|\xi|p}$ has the attractive and the repulsive Lagrangian submanifolds

$$\Lambda_-(0) = \left\{ (\pm\tfrac{\pi}{2}, x_2, \xi_1, 0) | \ x_2 \in \mathbb{S}^1, \ \pm\xi_1 > 0 \right\},$$

$$\Lambda_+(0) = \left\{ (\pm\tfrac{\pi}{2}, x_2, \xi_1, 0) | \ x_2 \in \mathbb{S}^1, \ \pm\xi_1 < 0 \right\}.$$

In Figure 2, we studied the numerical solution to (1.6) with

$$a = \tfrac{1}{2}, \ f(x) = -5\big(e^{-3((x_1+0.9)^2+(x_2+0.8)^2)} + e^{-3((x_1-0.9)^2+(x_2-0.8)^2)}\big)e^{2ix_1+ix_2}$$

and

$$\chi_0 \equiv 0, \ \chi_1(x) = \tfrac{1}{2}e^{-5(x_1-\pi/2)^2}, \chi_2(x) = \tfrac{1}{2}e^{-5(x_1-\pi/2)^2} + \tfrac{1}{2}e^{-5(x_1+\pi/2)^2}$$

for Figure 2(a)-2(c) respectively.

2. In (1.6), if $\mathbf{P} = P - i$, $f \in C^\infty(M)$, then from the spectral theory we know

$$u(t) = (P - i)^{-1}(e^{-t-itP} - I)f(x).$$

Since e^{-itP} preserves the L^2 norm and P is of 0th order, we know $u \in C^\infty([0, \infty); L^2(M))$ and

$$\|u(t)\|_{L^2} \leq (1 + e^{-t})\|f\|_{L^2}.$$

Moreover we have

$$\lim_{t\to\infty} u(t) = -\mathbf{P}^{-1}f(x) \text{ in } L^2(M).$$

2. MICROLOCAL CONTROL ESTIMATES

In this section, we prove Theorem 1. To simplify the notations, we assume $\omega = i\epsilon$, $\epsilon \geq 0$. The same proof works for ω such that $|\operatorname{Re}\omega| \leq \delta$, $\operatorname{Im}\omega \geq 0$, where δ satisfies the conditions in the remark in §1.2.

Let $\psi_t := e^{tH_{|\xi|p}} : T^*M \setminus 0 \to T^*M \setminus 0$ be the Hamiltonian flow of $|\xi|p$ on $T^*M \setminus 0$.

To state the following lemmata for coving $T^*M \setminus 0$ by the images of elliptic sets of operators under propagation, we recall that for a pseudodifferential operator $A \in \Psi^k(M)$, its elliptic set ell(A) is defined as a conic subset of $T^*M \setminus 0$, such that $(x_0, \xi_0) \in$ ell(A) if and only if there exists a conic neighborhood U of (x_0, ξ_0) in $T^*M \setminus 0$, such that

$$|\sigma(A)(x, \xi)| \geq C|\xi|^k, \ (x, \xi) \in U,$$

where $\sigma(A)$ is the principal symbol of A.

We then have

Lemma 2.1. *Suppose* A_-, $X_+ \in \Psi^0(M)$ *such that*

$$\Lambda_- \subset \text{ell}(A_-), \ \text{ell}(X_+) \text{ satisfies (CC+).}$$
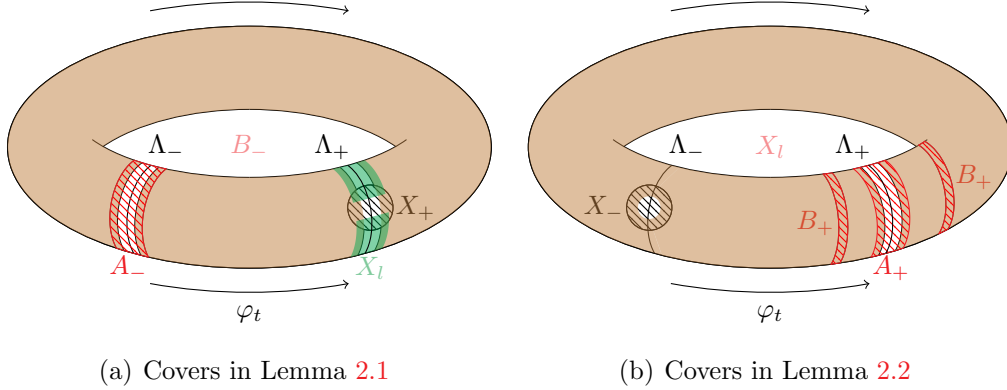
(a) Covers in Lemma 2.1

(b) Covers in Lemma 2.2

FIGURE 3. Illustration of the covers constructed in Lemma 2.1 and Lemma 2.2. The labels should be understood as the radial compactifications of the elliptic sets of the corresponding operators.

Then there exist finitely many operators $\{B_\ell, X_m\}_{\ell,m} \subset \Psi^0(M)$, *such that the elliptic sets of* A_-, B_ℓ, X_+, X_m *cover* $p^{-1}(0) \subset T^*M \setminus 0$ *and for each* ℓ, m, *there exist* $0 < T_\ell, T'_m < \infty$, *such that*

$$\mathrm{WF}(B_\ell) \subset \psi_{T_\ell}(\mathrm{ell}(A_-)), \ \mathrm{WF}(X_m) \subset \psi_{T'_m}(\mathrm{ell}(X_+)).$$

Proof. Since the bicharacteristics of $|\xi|p$ are the same as the ones for p in $T^*M \setminus 0$, it suffices to consider the radial compactification (by κ in §1.2) of all the conic sets in the lemma, and replace ψ_t by φ_t. We now identify the wavefront sets and elliptic sets for operators with their radial compactification.

Let Ω_+ be an open subset of $\mathrm{ell}(X_+)$ such that $\overline{\Omega_+} \subset \mathrm{ell}(X_+)$ and every connected component of γ_+ intersects Ω_+. Since γ_+ is compact, there exist finitely many T'_m, such that

$$\gamma_+ \subset U_+ := \bigcup_m \varphi_{T'_m}(\Omega_+).$$

Now let $U_- \subset T^*M \setminus 0$ be an open neighborhood of γ_- such that $\overline{U_-} \subset \mathrm{ell}(A_-)$.

Now for any $z \in \mathcal{S} \setminus U_+$, by (1.9), there exists $T(z)$ such that $z \in \varphi_{T(z)}(U_-)$. Notice that $\{\varphi_{T(z)}(U_-)\}_{z\in\mathcal{S}\setminus U_+}$ is an open cover of the compact set $\mathcal{S} \setminus U_+$, we can extract a finite cover $\{\varphi_{T(z_\ell)}(U_-)\}$ of $\mathcal{S} \setminus U_+$.

Let $b_\ell, \rho_m \in C^\infty(T^*M \setminus 0; \mathbb{R})$ be functions that are homogeneous of order 0 and

$$\mathrm{supp}\, b_\ell \subset \varphi_{T(z_\ell)}(\mathrm{ell}(A_-)), \ b_-|_{\overline{\varphi_{T(z_\ell)}(U_-)}} = 1; \ \mathrm{supp}\, \rho_m \subset \varphi_{T'_m}(\mathrm{ell}(X_+)), \ \rho_m|_{\overline{\varphi_{T'_m}(\Omega_+)}} = 1.$$

Put

$$B_\ell := \mathrm{Op}(b_\ell), \ X_m := \mathrm{Op}(\rho_m)$$

and we complete the proof. $\qquad\square$

Lemma 2.2. *Suppose A_+, B_+, $X_- \in \Psi^0(M)$ such that*

$$\Lambda_+ \subset \mathrm{ell}(A_+), \ \Lambda_+ \cap \mathrm{WF}(B_+) = \emptyset, \ \mathrm{ell}(X_-) \ \text{satisfies (CC$-$)}.$$

Then there exist finitely many operators $\{X_m\}_m \subset \Psi^0(M)$, such that the elliptic sets of A_+, X_-, X_m, cover $p^{-1}(0) \subset T^*M \setminus 0$, and the elliptic sets of X_-, X_m, cover $\mathrm{WF}(B_+)$. Moreover, for each m there exist $0 < T_m < \infty$, such that

$$\mathrm{WF}(X_m) \subset \psi_{T_m}(\mathrm{ell}(X_-)).$$

The proof for Lemma 2.2 is similar to the proof of Lemma 2.1.

We now prove the control estimates in Theorem 1.

Proof of Theorem 1. For $\omega = \lambda + i\epsilon$, $\lambda \in (-\delta, \delta)$, $\epsilon > 0$, we denote

$$\mathbf{P}_1(\omega) := \langle D \rangle^{1/2}(\mathbf{P} - \omega)\langle D \rangle^{1/2}, \ u_1 := \langle D \rangle^{-1/2}u \in C^\infty,$$

$$P_1 := \langle D \rangle^{1/2}(P - \lambda)\langle D \rangle^{1/2}, \ Q_1 := \langle D \rangle^{1/2}Q\langle D \rangle^{1/2}.$$

Principal type of propagation estimates. Suppose $A, B \in \Psi^0(M)$ such that there exists $T > 0$ such that

$$\psi_{-T}(\mathrm{WF}(B)) \subset \mathrm{ell}(A).$$

Then for any s, $N \in \mathbb{R}$, there exists $C > 0$ such that for any $u_1 \in C^\infty(M)$, we have

$$\|Bu_1\|_{H^s} \le C\|Au_1\|_{H^s} + C\|\mathbf{P}_1(\omega)u_1\|_{H^s} + C\|u_1\|_{H^{-N}}. \tag{2.1}$$

Here (2.1) follows from [DZ19a, Theorem E.47] and the fact that

$$\mathrm{Im}\,\sigma(\mathbf{P}_1(\omega)) = |\xi|(q - \epsilon) \le 0.$$

Radial source estimates. The goal of this step is to show the following estimate: there exists $A_- \in \Psi^0(M)$ such that $\Lambda_- \subset \mathrm{ell}(A_-)$ and for any $r, N \in \mathbb{R}$ such that

$$r > \sup_{\Lambda_-(\lambda)} \left(-\frac{|\xi|q_{-1}}{H_p|\xi|} \right),$$

there exists $C > 0$ such that for any $u_1 \in C^\infty(M)$, we have

$$\|A_- u_1\|_{H^r} \le C\|\mathbf{P}_1(\omega)u_1\|_{H^r} + C\|u_1\|_{H^{-N}}. \tag{2.2}$$

The proof of (2.2) is a modification of the proof of [DZ19a, Theorem E.52]. Indeed, let $B_- \in \Psi^0(M)$ such that $\Lambda_- \subset \mathrm{ell}(B_-)$, $\mathrm{WF}(B_-) \cap \Lambda_+ = \emptyset$ and $\rho \in C^\infty(T^*M \setminus 0; [0,1])$ be an *escape function* such that ρ is homogeneous of order 0 and

$$\mathrm{supp}\,\rho \subset \mathrm{ell}(B_-), \rho = 1 \ \text{near} \ \Lambda_-, \ H_{|\xi|(p-\lambda)}(\rho) \le 0.$$

For the construction of such functions, see [DZ16, Lemma C.1] or [DZ19a, Lemma E.53].

Now for $r \in \mathbb{R}$, we put $G := \mathrm{Op}(\langle\xi\rangle^r\rho) \in \Psi^r(M)$. Then

$$\mathrm{Im}\langle\mathbf{P}_1(\omega)u_1, G^*Gu_1\rangle = \langle-\tfrac{1}{2i}[P_1, G^*G]u_1, u_1\rangle + \langle(Q_1 - \epsilon\langle D\rangle)Gu_1, Gu_1\rangle$$
$$+ \langle\mathrm{Re}\,(G^*[G, Q_1 - \epsilon\langle D\rangle])\,u_1, u_1\rangle.$$

Notice that $\alpha\sqrt{-|\xi|q} \in S^{1/2}(T^*M; \mathbb{R})$, where α is given in §1.2. Let $Q_{1/2} := \mathrm{Op}(\alpha\sqrt{-|\xi|q}) \in \Psi^{1/2}(M)$, then

$$Q_1 + Q_{1/2}^*Q_{1/2} \in \Psi^0(M), \ \sigma(Q_1 + Q_{1/2}^*Q_{1/2}) = |\xi|q_{-1}.$$

Now we have

$$\mathrm{Im}\langle\mathbf{P}_1(\omega)u_1, G^*Gu_1\rangle$$
$$= \langle Tu_1, u_1\rangle - \|Q_{1/2}Gu_1\|_{L^2}^2 - \epsilon\|\langle D\rangle^{1/2}Gu_1\|_{L^2}^2 + \langle\mathrm{Re}(G^*[G, Q_1 - \epsilon\langle D\rangle])u_1, u_1\rangle \quad (2.3)$$
$$\leq \langle Tu_1, u_1\rangle + \langle\mathrm{Re}(G^*[G, Q_1 - \epsilon\langle D\rangle])u_1, u_1\rangle$$

with

$$T := -\tfrac{1}{2i}[P_1, G^*G] + G^*(Q_1 + Q_{1/2}^*Q_{1/2})G \in \Psi^{2r}(M), \ T^* = T.$$

(i) Since $G^*[G, Q_1 - \epsilon\langle D\rangle] \in \Psi^{2r}(M)$ has purely imaginary principal symbol, we know

$$\mathrm{Re}(G^*[G, Q_1 - \epsilon\langle D\rangle]) \in \Psi^{2r-1}(M), \ \mathrm{WF}(\mathrm{Re}(G^*[G, Q_1 - \epsilon\langle D\rangle])) \subset \mathrm{ell}(B_-).$$

By the elliptic estimates [DZ19a, Theorem E.33], we have

$$\langle\mathrm{Re}(G^*[G, Q_1 - \epsilon\langle D\rangle])u_1, u_1\rangle \leq C\|B_-u_1\|_{H^{r-1/2}}^2 + C\|u_1\|_{H^{-N}}^2. \quad (2.4)$$

(ii) The principal symbol of T is

$$\sigma(T) = |\xi|^{2r}\rho\left[H_{|\xi|(p-\lambda)}\rho + \rho\left(rH_p|\xi| + |\xi|q_{-1}\right)\right].$$

Let r satisfies

$$r > \max_{\Lambda_-(\lambda)}\left(-\frac{|\xi|q_{-1}}{H_p|\xi|}\right), \quad (2.5)$$

then there exists $\nu > 0$ and a conic open neighborhood $U \subset B_-$ of $\Lambda_-(\lambda)$ such that

$$rH_p|\xi| + |\xi|q_{-1} \leq -\nu \text{ on } U.$$

Thus

$$\sigma(T + \nu G^*G)|_U \leq 0.$$

Now apply the microlocal Gårding inequality [DZ19a, Proposition E.34], with (A, B, B_1) there replaced by $(-T - \nu G^*G, 0, B_-)$ and $h = 1$, we have

$$\nu\|Gu_1\|_{L^2}^2 \leq -\langle Tu_1, u_1\rangle + C\|B_-u_1\|_{H^{r-1/2}}^2 + C\|u_1\|_{H^{-N}}^2. \quad (2.6)$$

Combining (2.3), (2.4) and (2.6) and we find

$$\nu\|Gu_1\|_{L^2}^2 \leq -\mathrm{Im}\langle G\mathbf{P}_1(\omega)u_1, Gu_1\rangle + C\|B_-u_1\|_{H^{r-1/2}}^2 + C\|u_1\|_{H^{-N}}^2.$$

Use Cauchy–Schwartz and we have

$$\|Gu_1\|_{L^2} \leq C\|G\mathbf{P}_1(\omega)u_1\|_{L^2} + C\|B_-u_1\|_{H^{r-1/2}} + C\|u_1\|_{H^{-N}}.$$

Let $A_- := \mathrm{Op}(\rho)$, then by the elliptic estimates [DZ19a, Theorem E.33]

$$\|A_- u_1\|_{H^r} \le C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|B_- u_1\|_{H^{r-1/2}} + C\|u_1\|_{H^{-N}}. \tag{2.7}$$

To remove the $\|B_- u_1\|_{H^{r-1/2}}$ term, we use the propagation estimates (2.1) and find

$$\|B_- u_1\|_{H^{r-1/2}} \le C\|A_- u_1\|_{H^{r-1/2}} + C\|\mathbf{P}_1(\omega) u_1\|_{H^{r-1/2}} + C\|u_1\|_{H^{-N}}.$$

Apply the interpolation inequality for $H^{r-1/2}$ and H^r, H^{-N} to the term $\|A_- u_1\|_{H^{r-1/2}}$, and we have

$$\|B_- u_1\|_{H^{r-1/2}} \le \tfrac{1}{2}\|A_- u_1\|_{H^r} + C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|u_1\|_{H^{-N}}.$$

This together with (2.7) gives (2.2).

Radial sink estimates. Similar to (2.2), we can also prove the following radial sink estimates for $\mathbf{P}_1(\omega)$: there exist $A_+, B_+ \in \Psi^0(M)$ such that $\Lambda_+ \subset \mathrm{ell}(A_+)$, $\mathrm{WF}(B_+) \cap \Lambda_+ = \emptyset$ and for any $N \in \mathbb{R}$,

$$r < \inf_{\Lambda_+(\lambda)} \left(-\frac{|\xi| q_{-1}}{H_p |\xi|} \right)$$

there exists $C > 0$ such that for any $u_1 \in C^\infty$, we have

$$\|A_+ u_1\|_{H^r} \le C\|B_+ u_1\|_{H^r} + C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|u_1\|_{H^{-N}}. \tag{2.8}$$

Control by the sink. Suppose $X_+ \in \Psi^0(M)$ and $\mathrm{ell}(X_+)$ satisfies (CC+). We now prove (1.2). Without loss of generality, we assume $\mathrm{Re}(\omega) = 0$.

Let $\widetilde{X}_+ \in \Psi^0(M)$ be such that $\mathrm{WF}(\widetilde{X}_+) \subset \mathrm{ell}(X_+)$ and $\mathrm{ell}(\widetilde{X}_+)$ satisfy (CC+). Let $A_- \in \Psi^0(M)$ be as in (2.2). Then A_-, $\widetilde{X}_+$ satisfy conditions in Lemma 2.1 and we can find B_ℓ, $X_m \in \Psi^0(M)$ satisfying conditions in Lemma 2.1. Then by (2.1), for any $r, N \in \mathbb{R}$ and each ℓ, m, there exists $C > 0$ such that for any $u_1 \in C^\infty$, we have

$$\begin{aligned} \|B_\ell u_1\|_{H^r} &\le C\|A_- u_1\|_{H^r} + C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|u_1\|_{H^{-N}}, \\ \|X_m u_1\|_{H^r} &\le C\|\widetilde{X}_+ u_1\|_{H^r} + C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|u_1\|_{H^{-N}}. \end{aligned} \tag{2.9}$$

Combine (2.2), (2.9), the fact that the elliptic sets of A_-, B_ℓ, $\widetilde{X}_+$, X_m covers $p^{-1}(0)$, and the elliptic estimates [DZ19a, Theorem E.33], we know that for r satisfying (2.5)

$$\|u_1\|_{H^r} \le C\|\widetilde{X}_+ u_1\|_{H^r} + C\|\mathbf{P}_1(\omega) u_1\|_{H^r} + C\|u_1\|_{H^{-N}}.$$

The control estimate (1.2) now follows from the elliptic estimate applied to $\widetilde{X}_+ \langle D \rangle^{\frac{1}{2}}$ and $\langle D \rangle^{\frac{1}{2}} X_+$.

Control by the source. The control estimate (1.4) is proved in a similar way.

Assume $X_- \in \Psi^0(M)$ satisfies (CC−). Let $\widetilde{X}_- \Psi^0(M)$ be such that $\mathrm{WF}(\widetilde{X}_-) \subset \mathrm{ell}(X_-)$ and $\mathrm{ell}(\widetilde{X}_-)$ satisfy (CC−). Let $A_+, B_+ \in \Psi^0(M)$ be as in (2.8). We then apply Lemma 2.2 to A_+, B_+, $\widetilde{X}_-$ and get operators X_m.

By (2.1) we find for r, $N \in \mathbb{R}$,

$$\|X_m u_1\|_{H^r} \leq C\|\widetilde{X}_- u_1\|_{H^r} + C\|\mathbf{P}_1(\omega)u_1\|_{H^r} + C\|u_1\|_{H^{-N}}. \tag{2.10}$$

The estimate (1.4) is now a result of (2.8), (2.10) and elliptic estimates. $\qquad\square$

3. The limiting absorption principle

From now on we assume

$$\mathbf{P} = P - i\chi, \ \chi \in C^\infty(M; \mathbb{R}_{\geq 0}). \tag{3.1}$$

We ask $T^*(\operatorname{supp}\chi) \setminus 0$ to satisfy the control condition (CC+). This is equivalent to say that

$$\operatorname{supp}\chi \text{ intersects with each connected component of } \pi(\Lambda_+). \tag{3.2}$$

Here $\pi : T^*M \setminus 0 \to M$ is the natural projection. We remark that in this case, $q_{-1} = 0$. Hence the regularity conditions (1.1), (1.3) become

$$s > -\tfrac{1}{2}, \text{ or } s < -\tfrac{1}{2}$$

respectively.

Lemma 3.1. *Suppose* $\mathbf{P}$ *satisfies* (1.8), (1.9), (3.1), (3.2). *Then there exists* $\delta > 0$ *such that*

$$|\operatorname{Spec}_{L^2, \mathrm{pp}}(\mathbf{P}) \cap [-\delta, \delta]| < \infty.$$

Proof. By [DZ19b, Lemma 3.2], it suffices to show that

$$\{\operatorname{Re}\omega \mid \omega \in \operatorname{Spec}_{L^2, \mathrm{pp}}(\mathbf{P}), \operatorname{Im}\omega \geq 0\} \subset \operatorname{Spec}_{L^2, \mathrm{pp}}(P).$$

In fact, assume $\omega \in \mathbb{C}$, $\operatorname{Im}\omega \geq 0$, $u \in \operatorname{Ker}_{L^2}(\mathbf{P} - \omega)$. Then

$$0 = \operatorname{Im}\langle(\mathbf{P} - \omega)u, u\rangle = -\langle(\chi + \operatorname{Im}\omega)u, u\rangle = -\int_M (\chi + \operatorname{Im}\omega)|u|^2 dx.$$

Since $\chi + \operatorname{Im}\omega \geq 0$, we know $(\chi + \operatorname{Im}\omega)u = 0$. Thus $(P - \operatorname{Re}\omega)u = 0$. $\qquad\square$

We record a lemma on the regularity of solutions.

Lemma 3.2. *Let* $\mathbf{P}$ *be as in Lemma 3.1. Then there exists* $\delta > 0$ *such that*

$$(\mathbf{P} - \omega)u \in C^\infty, \ u \in \mathscr{D}'(M), \ \omega \in [-\delta, \delta]$$

implies that $u \in C^\infty(M)$.

Proof. Let $\widetilde{\chi} \in C^\infty(M)$ satisfy $\operatorname{supp}\widetilde{\chi} \Subset \{\chi > 0\}$ and (3.2). Then notice that

$$\sigma(\mathbf{P} - \omega) = p - \omega - i\chi,$$

which is elliptic on $\operatorname{supp}\widetilde{\chi}$. Hence by the elliptic estimate, for $s, N \in \mathbb{R}$, there exists $C > 0$, such that

$$\|\widetilde{\chi}u\|_{H^s} \leq C\|(\mathbf{P} - \omega)u\|_{H^s} + C\|u\|_{H^{-N}}.$$

This implies $\widetilde{\chi}u \in C^\infty(M)$. The desired result $u \in C^\infty(M)$ now follows from (1.2) with X_+ replaced by $\widetilde{\chi}$. $\square$

Now we show the limiting absorption principle:

Lemma 3.3. *Suppose* **P** *is as in Lemma 3.1 and* $0 \notin \mathrm{Spec}_{\mathrm{pp}}(\mathbf{P})$. *Then there exists* $\delta > 0$ *such that for any* $\omega \in [-\delta, \delta]$, $f \in C^\infty(M)$, *the limit*

$$(\mathbf{P} - \omega - i0)^{-1}f := \lim_{\epsilon \to 0+}(\mathbf{P} - \omega - i\epsilon)^{-1}f$$

exists in $C^\infty(M)$. *Moreover,*

$$(\mathbf{P} - \omega - i0)^{-1}f \in C_\omega^\infty((-\delta, \delta); C^\infty(M))$$

and is the unique solution to the equation

$$(\mathbf{P} - \omega)u = f, \ u \in \mathscr{D}'(M).$$

Proof of Lemma 3.3. 1. *The limit exists and is smooth on* M. By Lemma 3.1, for $\delta > 0$ sufficiently small, we have $\mathrm{Spec}_{\mathrm{pp}}(\mathbf{P}) \cap [-\delta, \delta] = \emptyset$. We first show that for any $\omega \in [-\delta, \delta]$, $f \in C^\infty(M)$, we have

$$(\mathbf{P} - \omega - i0)^{-1}f \in C^\infty(M).$$

For simplicity, we put $\omega = 0$.

For any $\epsilon > 0$, since $\chi \geq 0$, we know $\mathbf{P} - i\epsilon$ is elliptic. Thus $u_\epsilon := (\mathbf{P} - i\epsilon)^{-1}f \in C^\infty(M)$. Let $\widetilde{\chi}$ be as in the proof of Lemma 3.2. Theorem 1 implies that for any $s > -\frac{1}{2}$, $N \in \mathbb{R}$, there exists $C > 0$ such that for any $\epsilon > 0$,

$$\|u_\epsilon\|_{H^s} \leq C\|\widetilde{\chi}u_\epsilon\|_{H^s} + C\|f\|_{H^{s+1}} + C\|u_\epsilon\|_{H^{-N}}. \tag{3.3}$$

On the other hand, for $\epsilon > 0$, we have

$$\mathrm{Im}\,\sigma(\mathbf{P} - i\epsilon) = -(\chi + \epsilon),$$

which is uniformly elliptic over $\mathrm{supp}\,\widetilde{\chi}$. Thus by the elliptic estimates [DZ19a, Theorem E.33], for any s, $N \in \mathbb{R}$, there exists $C > 0$ such that

$$\|\widetilde{\chi}u_\epsilon\|_{H^s} \leq C\|f\|_{H^s} + C\|u_\epsilon\|_{H^{-N}}. \tag{3.4}$$

Estimates (3.3), (3.4) implies that for any $s > -\frac{1}{2}$,

$$\|u_\epsilon\|_{H^s} \leq C\|f\|_{H^{s+1}} + C\|u_\epsilon\|_{H^{-N}}. \tag{3.5}$$

We now show that for any $s > -\frac{1}{2}$, $\{\|u_\epsilon\|_{H^s}\}_{\epsilon>0}$ is bounded. Otherwise, there exists $s > -\frac{1}{2}$ and a subsequence u_ℓ such that $\|u_\ell\|_{H^s} \to \infty$, $\ell \to \infty$. We put $\widetilde{u}_\ell := u_\ell/\|u_\ell\|_{H^s}$, then

$$\|\widetilde{u}_\ell\|_{H^s} = 1, \ (\mathbf{P} - i\epsilon)\widetilde{u}_\epsilon = f/\|u_\ell\|_{H^s} \xrightarrow{C^\infty} 0, \ \ell \to \infty. \tag{3.6}$$

By (3.5), for any $s' > -\frac{1}{2}$, we have

$$\|\widetilde{u}_\ell\|_{H^{s'}} \leq C\|f\|_{H^{s'+1}} + C\|\widetilde{u}_\ell\|_{H^s} = C\|f\|_{H^{s'+1}} + C.$$

In particular, we know $\widetilde{u}_\ell$ is bounded in $H^{s'}$ for $s' > s$. Since the embedding $H^{s'} \hookrightarrow H^s$ is compact when $s' > s$, by passing to a subsequence, we can assume that $u_\ell \to u$ in H^s. Let $\ell \to \infty$ in (3.6), we find

$$\|u\|_{H^s} = 1, \ \mathbf{P}u = 0.$$

This contradicts Lemma 3.2 and the assumption $0 \notin \mathrm{Spec}_{\mathrm{pp}}(\mathbf{P})$.

We conclude now that $\{u_\epsilon\}_{\epsilon>0}$ is bounded in H^s for any $s > -\frac{1}{2}$ and hence in H^s for any $s \in \mathbb{R}$. A similar argument as above shows that $\{u_\epsilon\}_{\epsilon>0}$ is precompact in H^s for any $s \in \mathbb{R}$. Notice that by $(\mathbf{P} - i\epsilon)u_\epsilon = f$ and (3.5), every limit point u has to satisfy

$$\mathbf{P}u = f, \ u \in \mathscr{D}'(M). \tag{3.7}$$

By either (3.5) or Lemma 3.2, we can see that $u \in C^\infty$. By Lemma 3.2, we know such u is unique. Hence u_ϵ converges to the unique solution to (3.7).

2. *Smoothness in ω.* First, to see that $(\mathbf{P} - \omega - i0)^{-1}f$ is continuous in ω for $\omega \in (-\delta, \delta)$, we can replace $u_\epsilon = (\mathbf{P} - i\epsilon)^{-1}f$ above by $u_\ell := (\mathbf{P} - \omega_\ell - i\epsilon_\ell)^{-1}f$ with

$$\epsilon_\ell > 0, \ \omega_\ell \in (-\delta, \delta), \ \omega_\ell + i\epsilon_\ell \to \omega \in (-\delta, \delta).$$

The previous argument shows that $\{u_\ell\}$ converges in H^s for any s to the unique solution to

$$u \in C^\infty, \ (\mathbf{P} - \omega)u = f,$$

which is $(\mathbf{P} - \omega - i0)^{-1}f$. This implies $(\mathbf{P} - \omega - i0)^{-1}f$ is continuous in $\omega \in (-\delta, \delta)$.

For any $k \in \mathbb{N}$, we denote

$$(\mathbf{P} - \omega - i0)^{-k}f := \left((\mathbf{P} - \omega - i0)^{-1}\right)^k f.$$

Then we have $(\mathbf{P} - \omega - i0)^{-k}f \in C^\infty$. We claim that for $\omega \in (-\delta, \delta)$,

$$\partial_\omega^k \left((\mathbf{P} - \omega - i0)^{-1}f\right) = k!(\mathbf{P} - \omega - i0)^{-k-1}f, \ k \geq 1. \tag{3.8}$$

In fact, for $k = 1$, $\omega_0 \in (-\delta, \delta)$ we have

$$\frac{(\mathbf{P} - \omega - i0)^{-1}f - (\mathbf{P} - \omega_0 - i0)^{-1}f}{\omega - \omega_0} = (\mathbf{P} - \omega - i0)^{-1}(\mathbf{P} - \omega_0 - i0)^{-1}f.$$

Let $\omega \to \omega_0$ and use the continuity in ω, and we find

$$\partial_\omega|_{\omega=\omega_0}(\mathbf{P} - \omega - i0)^{-1}f = (\mathbf{P} - \omega_0 - i0)^{-2}f.$$

(3.8) then follows by induction in k. This concludes the proof. $\qquad\qquad\square$

4. The damped equation

In this section we study the evolution problem for $\mathbf{P} = P - i\chi$, which considered as the damping problem for P and prove Theorem 2.

Proof of Theorem 2. Since $\mathbf{P} : L^2(M) \to L^2(M)$ is bounded, we define

$$U(t) := \sum_{\ell=0}^{\infty} \frac{(-it\mathbf{P})^{\ell}}{\ell!} : L^2(M) \to L^2(M), \ t \in \mathbb{R}.$$

Then $U(t)$ is also bounded for any $t \in \mathbb{R}$.

We first show that (1.6) has a unique solution

$$u(t) := (U(t) - I)(\mathbf{P} - i0)^{-1}f.$$

Indeed, by Lemma 3.3, we have $(\mathbf{P} - i0)^{-1}f \in C^{\infty} \subset L^2(M)$. We thus have $u(t) \in L^2(M)$ and can check that

$$(i\partial_t - \mathbf{P})u(t) = f, \ u(0) = 0.$$

Suppose (1.6) has another solution $w(t)$, then we have

$$(i\partial_t - \mathbf{P})(u - w) = 0, \ u(0) - w(0) = 0.$$

Now we compute

$$\begin{aligned} 0 =& 2\operatorname{Im}\langle (i\partial_t - \mathbf{P})(u - w), u - w \rangle_{L^2(M)} \\ =& \ \partial_t \|u - w\|^2_{L^2(M)} + 2\langle \chi(u - w), u - w \rangle_{L^2(M)} \geq \partial_t \|u - w\|^2_{L^2(M)}. \end{aligned}$$

Thus we know

$$\|u(t) - w(t)\|_{L^2(M)} \leq \|u(0) - w(0)\|_{L^2(M)} = 0 \ \Rightarrow \ u = w, \ t \geq 0.$$

To see that $\|u(t)\|_{L^2(M)}$ is uniformly bounded in t, we notice that

$$(i\partial_t - \mathbf{P})(u(t) + (\mathbf{P} - i0)^{-1}f) = 0.$$

We again compute

$$0 = \operatorname{Im}\langle (i\partial_t - \mathbf{P})(u + (\mathbf{P} - i0)^{-1}f), u + (\mathbf{P} - i0)^{-1}f \rangle \geq \partial_t \|u + (\mathbf{P} - i0)^{-1}f\|^2_{L^2(M)}.$$

Thus

$$\|u + (\mathbf{P} - i0)^{-1}f\|_{L^2(M)} \leq \|(\mathbf{P} - i0)^{-1}f\|_{L^2(M)},$$

which implies

$$\|u(t)\|_{L^2(M)} \leq 2\|(\mathbf{P} - i0)^{-1}f\|_{L^2(M)}.$$

This concludes the proof of Theorem 2. □

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