NEUMANN DATA MASS ON PERTURBED TRIANGLES

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Abstract. Based on a previous paper [Chr17] on Neumann data for Dirichlet eigenfunctions on triangles, we extend the study in two ways. First, we investigate the (semi-classical) Neumann data mass on perturbed triangles. Specifically, we replace one side of a triangle by adding a smooth perturbation, and assume that the disparity between the perturbation and the original side is bounded by a small value ϵ. Second, we add a small ϵ sized potential to the (semi-classical) Laplacian and see how the results change on triangles. In both cases, we find that the L^2 norm of Neumann data on each side is close to the length of the side divided by the area of the triangle, and the difference is dominated by ϵ.

1. Introduction

The purpose of this paper is to extend the results on triangles in [Chr17] by the first author to small perturbations, both in the domain and in the Laplacian.

In our first extension, we study the Dirichlet eigenfunction problem in a new domain that is modified from a triangle. Given a bounded domain $D \subset \mathbb{R}^2$, consider the Dirichlet eigenfunction problem:

$$\begin{cases} (-h^2 \triangle - 1)u = 0 \text{ in } D \\ u|_{\partial D} = 0 \end{cases} \tag{1.1}$$

where the eigenfunctions are assumed to be normalized $\|u\|_{L^2(D)} = 1$.

In [Chr17] by the first author, it is shown that if the domain D is a triangle, the Neumann data mass is equally distributed on each face: If F_1, F_2, and F_3 are the three faces of T with lengths l_1, l_2, and l_3 respectively, then

$$\int_{F_j} |h\partial_\nu u|^2 dS = \frac{l_j}{\text{Area}(D)}, \quad j = 1, 2, 3,$$

where $h\partial_\nu u$ is the semi-classical normal derivative on ∂T, dS is the arclength measure, Area(D) is the area of the triangle D. In other words, the L^2 norm of Neumann data on each side equals to the length of the side divided by the area of the triangle. An analogous result holds [Chr18] when the dimension $n \geq 3$, so the purpose of the present paper is to extend the 2 dimensional results to other domains.

To look into the Neumann data mass in different domains, we first study domains that are close to triangles: we construct a planar domain by changing one side of a triangle to a smooth function which is close to linear in a suitable sense. Moreover, we restrict this function to be close to the original side of the triangle, and their disparity is bounded by a small number ϵ. Our main findings show that the Neumann data mass on each side is close to that of the original triangle, and the difference is dominated by ϵ.

Let $T \subset \mathbb{R}^2$ be a triangle with sides A, B, and C with lengths a, b, and c respectively. Assume the triangle is oriented so that one corner is at the origin, side A is vertical, and side C is parametrized by $a_2 x/l$, $0 \leq x \leq l$, for positive a_2 and l (see Figure 1 for a picture in the acute case and Figure 2 for the obtuse case). Let $\tilde{g}(x)$ be a smooth function satisfying $\tilde{g}(0) = \tilde{g}(l) = 0$, $|\tilde{g}(x)| \leq 1$, and $|\tilde{g}'(x)| \leq 1$. For $\epsilon > 0$ small, let $g_\epsilon(x) = \epsilon\tilde{g}(x)$. Let $D_\epsilon \subset \mathbb{R}^2$ be the domain with side C replaced by side C' parametrized by $a_2 x/l + g_\epsilon(x)$.

Theorem 1. *Fix $\epsilon > 0$ small and let $D_\epsilon \subset \mathbb{R}^2$ be the domain described above, and suppose $\{u_h^\epsilon\}_h$ solves the semiclassical eigenfunction problem* (1.1) *with D replaced by D_ϵ. Then*

$$\int_A |h\partial_\nu u^\epsilon|^2 dS = \frac{a}{\mathit{Area}(D)} + \mathcal{O}(\epsilon)$$

$$\int_B |h\partial_\nu u^\epsilon|^2 dS = \frac{b}{\mathit{Area}(D)} + \mathcal{O}(\epsilon),$$

and

$$\int_{C'} |h\partial_\nu u^\epsilon|^2 dS = \frac{l(C')}{\mathit{Area}(D)} + O(\epsilon),$$

where $l(C')$ is the length of side C'.

Remark 1.1. The theorem states that the equidistribution law from [Chr17] is stable under small perturbations. The implicit constants are independent of $\tilde{g}$ as long as it satisfies $|\tilde{g}| \leq 1$ and $|\tilde{g}'| \leq 1$. In fact, we really only use that $|g| \leq \epsilon$ and $|g'| \leq \epsilon$ in the proof, so g does not have to be of the form $\epsilon\tilde{g}$, however it does make the statement of the theorem more clear.

Remark 1.2. We expect that the analogue of Theorem 1 (and Theorem 2 below) hold in higher dimensions, following the work [Chr18] by the first author.

In our second extension, we consider a modified Dirichlet eigenfunction problem on triangles.

Theorem 2. *Let $T \subset \mathbb{R}^2$ be a triangle with sides A, B, and C with lengths a, b, and c respectively. Let $\tilde{w}(x, y)$ be a smooth function on T with $|\tilde{w}| \leq 1$ and $|\nabla\tilde{w}| \leq 1$. For $\epsilon > 0$ small, let $w_\epsilon = \epsilon\tilde{w}$. Consider the eigenfunction problem*

$$\begin{cases} -h^2 \triangle + w_\epsilon(x, y))u^\epsilon = u^\epsilon \text{ on } T, \\ u^\epsilon|_{\partial_T} = 0, \end{cases}$$

and assume the u^ϵ are normalized $\|u^\epsilon\|_{L^2(T)} = 1$. Then for $\epsilon > 0$ sufficiently small

$$\int_A |h\partial_\nu u^\epsilon|^2 dS = \frac{a}{\mathit{Area}(T)} + \mathcal{O}(\epsilon)$$

$$\int_B |h\partial_\nu u^\epsilon|^2 dS = \frac{b}{\mathit{Area}(T)} + \mathcal{O}(\epsilon),$$

and

$$\int_C |h\partial_\nu u^\epsilon|^2 dS = \frac{c}{\mathit{Area}(T)} + O(\epsilon).$$

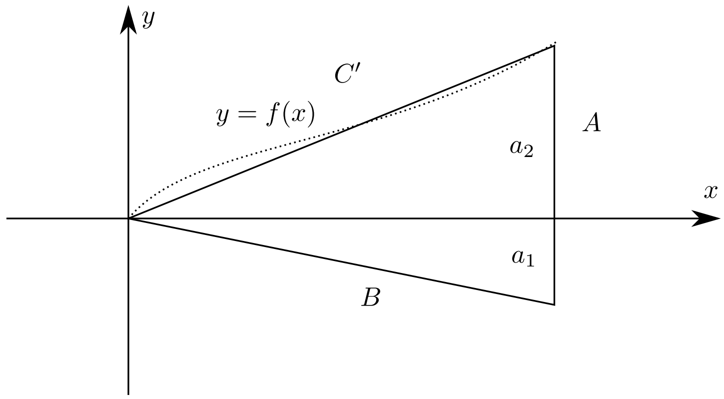

FIGURE 1. Setup for acute (and right) triangles.

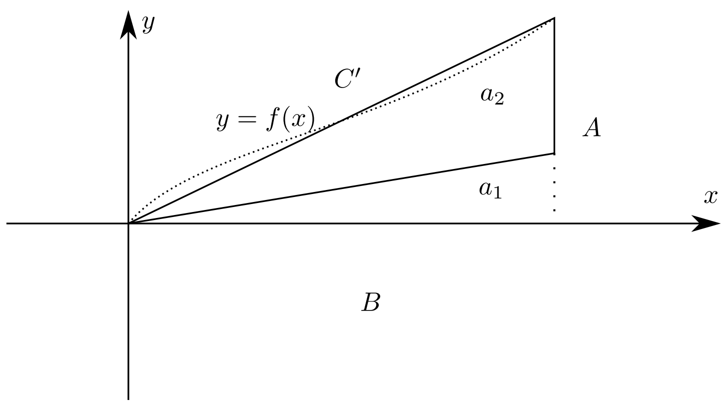

FIGURE 2. Setup for obtuse triangles.

1.1. **History.** The study of restrictions of eigenfunctions and the study of boundary traces is an old subject. In this very abbreviated history we just focus on some of the recent developments particularly relevant to the present work. Previous results on restrictions primarily focused on upper bounds. In general, it is difficult to separate the behaviour of the Dirichlet and Neumann data for restrictions to interior hypersurfaces. In the paper of Burq-Gérard-Tzvetkov [BGT07], restrictions of the Dirichlet data to arbitrary smooth hypersurfaces on manifolds without boundary were considered. An upper bound of the norm (squared) of the restrictions of $\mathcal{O}(h^{-1/2})$ was proved, and shown to be sharp. Of course this sharpness shows that there are *some* eigenfunctions with a known lower bound. In the first author's paper with Hassell-Toth [CHT15], an upper bound of $\mathcal{O}(1)$ was proved for (semi-classical) Neumann data restricted to arbitrary smooth hypersurfaces on manifolds without boundary, and this also shown to be sharp. Again, this gives a lower and upper bound for the Neumann data alone for *some* eigenfunctions.

In the case of quantum ergodic eigenfunctions, more is known. In the papers of Gérard-Leichtnam [GL93] and Hassell-Zelditch [HZ04], the Neumann (respectively Dirichlet) boundary data of Dirichlet (respectively Neumann) quantum ergodic eigenfunctions is studied, and shown to have an asymptotic formula for a *density one* subsequence. Similar statements were proved for interior hypersurfaces by Toth-Zelditch [TZ12, TZ13]. Again, potentially a sparse subsequence may behave differently. In the author's paper with Toth-Zelditch [CTZ13], an asymptotic formula for the whole weighted Cauchy data is proved for the entire sequence of quantum ergodic eigenfunctions, however it is impossible to separate the behaviour of the Dirichlet versus Neumann data. And the sequence of eigenfunctions is assumed to already be quantum ergodic, having thrown out any weird sparse subsequences.

In [Chr17] and [Chr18], the first author studied the Neumann boundary data for Dirichlet eigenfunctions on simplices and proved an equidistribution law. It is not an asymptotic, but an exact identity, and holds for the entire sequence of eigenfunctions. It agrees with what the paper of Hassell-Zelditch would give, but is an exact identity and holds for the whole sequence of eigenfunctions, so hints at quantum ergodicity (or at least some weak form of quantum ergodicity). The main purpose of this paper is to study similar phenomena for eigenfunction problems which are "close to" triangle eigenfunction problems.

2. Proof of Theorem 1

For the rest of the paper, let us drop the awkward u^ϵ notation and just write u, being careful to keep in mind that everything implicitly depends on ϵ.

We first need to prove that the Neumann data on side C' is still bounded independent of ϵ, since our $\mathcal{O}(\epsilon)$ error estimates are in terms of a priori Neumann data estimates on side C'.

2.1. The upper bound of $\int_{C'} |h\partial_\nu u|^2 dS$.

One of the assumptions we will use is that $\int_{C'} |h\partial_\nu u|^2 dS$ is bounded by a number that is independent of ϵ.

Lemma 2.1. *Let $D \subset \mathbb{R}^2$ be the domain from Theorem 1. Then for $\epsilon > 0$ sufficiently small, there exists a number Γ, independent of h and ϵ so that*

$$\int_{C'} |h\partial_\nu u|^2 dS \leq \Gamma.$$

The proof of Lemma 2.1 is in Section 4 after the proofs of Theorems 1 and 2.

2.2. Proof of Theorem 1 for acute triangles.

As shown in Figure 1, sides A, B, C' are listed in clockwise orientation. We use rectangular coordinates (x, y) and orient our triangle such that the corner between sides B and C' is at the origin $(0, 0)$, and the side A is parallel to the y axis.

Let l be the segment on the x axis that begins at $(0, 0)$ and is perpendicular to the side A. Write $A = A_1 \cup A_2$, where A_1 as the part of A below the x axis, and A_2 as the part above the x axis. Assume a_1 and a_2 be their lengths. We modify the acute triangle by replacing its original side C as described in the theorem.

Specifically, we can parametrize side B and C' with respect to x:

$$B = \{(x, y) \in \mathbb{R}^2 : y = -\frac{a_1}{l}x, 0 \leq x \leq l\}$$

and

$$C' = \{(x,y) \in \mathbb{R}^2 : f(x) = \frac{a_2}{l}x + g(x), 0 \leq x \leq l\}$$

where $g(x)$ is restricted by the two conditions from the theorem so that the function $f(x)$ is close to the original side of the acute triangle:

$$|g(x)| \leq \epsilon$$

$$|g'(x)| \leq \epsilon.$$

Then the arclength parameters are $\gamma_A = 1$,

$$\gamma_B = \left(1 + (\frac{a_1}{l})^2\right)^{\frac{1}{2}} = \frac{(l^2 + a_1^2)^{\frac{1}{2}}}{l} = \frac{b}{l}$$

and

$$\gamma_{C'} = (1 + (f'(x))^2)^{\frac{1}{2}}$$

We can then derive the unit tangent vectors:

$$\tau_A = (0, 1),$$

$$\tau_B = \left(1, -\frac{a_1}{l}\right)\gamma_B^{-1} = \left(\frac{l}{b}, -\frac{a_1}{b}\right)$$

and

$$\tau_{C'} = (1, f'(x))\gamma_{C'}^{-1} = \left(\frac{1}{\sqrt{1 + (f'(x))^2}}, \frac{f'(x)}{\sqrt{1 + (f'(x))^2}}\right).$$

From the unit tangent vectors, we find the outward unit normal vectors to be:

$$\nu_A = (1, 0),$$

$$\nu_B = \left(-\frac{a_1}{b}, -\frac{l}{b}\right)$$

and

$$\nu_{C'} = \left(-\frac{f'(x)}{\sqrt{1 + (f'(x))^2}}, \frac{1}{\sqrt{1 + (f'(x))^2}}\right)$$

The Dirichlet boundary conditions imply that the tangential derivatives of u vanish on the boundary of the domain. That is

$$\partial_y u = 0$$

on A,

$$\tau_B \cdot \nabla u = \frac{l}{b}\partial_x u - \frac{a_1}{b}\partial_y u = 0$$

on B, and

$$\tau_{C'} \cdot \nabla u = \frac{1}{\sqrt{1 + (f'(x))^2}}\partial_x u + \frac{f'(x)}{\sqrt{1 + (f'(x))^2}}\partial_y u = 0$$

on C'. Rearranging, we get

$$h\partial_x u = \frac{a_1}{l}h\partial_y u$$

on B, and

$$h\partial_x u = -f'(x)h\partial_y u$$

on C'.

Next, we can relate ∂_x and ∂_y to ∂_ν on each side. Along B, we have

$$\begin{aligned} h\partial_{\nu_B} u &= \nu_B \cdot h\nabla u \\ &= -\frac{a_1}{b} h\partial_x u - \frac{l}{b} h\partial_y u \\ &= \left(-\frac{a_1^2}{bl} - \frac{l}{b}\right) h\partial_y u \\ &= -\frac{b}{l} h\partial_y u \end{aligned}$$

and thus

$$\begin{aligned} h\partial_y u &= -\frac{l}{b} h\partial_{\nu_B} u \\ h\partial_x u &= -\frac{a_1}{b} h\partial_{\nu_B} u \end{aligned}$$

Similarly, along C', we have

$$\begin{aligned} h\partial_{\nu_{C'}} u &= \nu_{C'} \cdot h\nabla u \\ &= -\frac{f'(x)}{\sqrt{1+(f'(x))^2}} h\partial_x u + \frac{1}{\sqrt{1+(f'(x))^2}} h\partial_y u \\ &= (\frac{(f'(x))^2+1}{\sqrt{1+(f'(x))^2}}) h\partial_y u \end{aligned}$$

Hence along C' we have

$$\begin{aligned} h\partial_y u &= \frac{1}{\sqrt{1+(f'(x))^2}} h\partial_{\nu_{C'}} u \\ h\partial_x u &= -\frac{f'(x)}{\sqrt{1+(f'(x))^2}} h\partial_{\nu_{C'}} u \end{aligned}$$

Now consider the operator

$$X = (x+m)\partial_x + (y+n)\partial_y$$

where m, n are parameters that are independent of x and y. The usual computation yields

$$[-h^2 \triangle - 1, X] = -2h^2 \triangle$$

Then using the eigenfunction equation we have

$$\begin{aligned} \int_D ([-h^2 \triangle - 1, X]u)\overline{u} dV &= -2 \int_D (h^2 \triangle u)\overline{u} dV \\ &= \int_D 2|u|^2 dV \\ &= 2 \end{aligned}$$

since u is normalized to the length of one.

Another way of calculation is

$$\begin{aligned}
&\int_D ([-h^2\triangle - 1, X]u)\overline{u}dV \\
&= \int_D ((-h^2\triangle - 1)Xu)\overline{u}dV - \int_D (X(-h^2\triangle - 1)u)\overline{u}dV \\
&= \int_D ((-h^2\triangle - 1)Xu)\overline{u}dV,
\end{aligned}$$

where we have used the eigenfunction equation in the last line.

Integrating by parts and applying the Green's theorem, we get

$$\begin{aligned}
\int_D ((-h^2\triangle - 1)Xu)\overline{u}dV &= \int_D (Xu)(-h^2\triangle - 1)\overline{u}dV \\
&\quad - \int_{\partial D} (h\partial_\nu hXu)\overline{u}dS + \int_{\partial D} (hXu)(h\partial_\nu \overline{u})dS \\
&= \int_{\partial D} (hXu)(h\partial_\nu \overline{u})dS,
\end{aligned}$$

where we have used the Dirichlet boundary conditions in the last line.

Combining the results we have

$$2 = \int_{\partial D} (hXu)(h\partial_\nu \overline{u})dS$$

which we can integrate on three sides separately.

To simplify the notation, we define

$$I_A = \int_A |h\partial_\nu u|^2 dS$$

and similarly for B and C'.

Along A, we have

$$\begin{aligned}
&\int_A (hXu)(h\partial_{\nu_A}\overline{u})dS \\
&= \int_A (((x+m)h\partial_x + (y+n)h\partial_y)u)(h\partial_{\nu_A}\overline{u})dS \\
&= (l+m)I_A
\end{aligned}$$

where $x = l$ on the side A.

On the side B, we substitute $y = -\frac{a_1}{l}x$ and get

$$\begin{aligned}
&\int_B (hXu)(h\partial_{\nu_B}\overline{u})dS \\
&= \int_B \left(\left((x+m)h\partial_x + (-\frac{a_1}{l}x + n)h\partial_y\right)u\right)(h\partial_{\nu_B}\overline{u})dS \\
&= \int_B \left(\left(\left((x+m)\left(-\frac{a_1}{b}\right) + \left(-\frac{a_1}{l}x + n\right)\left(-\frac{l}{b}\right)\right)h\partial_{\nu_B}\right)u\right)(h\partial_{\nu_B}\overline{u})dS \\
&= \int_B \left(\left(-\frac{a_1}{b}m - \frac{l}{b}n\right)h\partial_{\nu_B}u\right)(h\partial_{\nu_B}\overline{u})dS \\
&= \left(-\frac{a_1}{b}m - \frac{l}{b}n\right)I_B
\end{aligned}$$

On C', we substitute $y = f(x)$ and get

$$\begin{aligned}
&\int_{C'} (hXu)(h\partial_{\nu_{C'}}\overline{u})dS \\
&= \int_{C'} (((x+m)h\partial_x + (y+n)h\partial_y)u)(h\partial_{\nu_{C'}}\overline{u})dS \\
&= \int_{C'} \left(\left(-(x+m)\frac{f'(x)}{\sqrt{1+f'(x)^2}} + (f(x)+n)\frac{1}{\sqrt{1+f'(x)^2}}\right)\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS
\end{aligned}$$

Hence, summing up the integrations along the three sides, we have

$$\begin{aligned}
&\int_{\partial D} (hXu)(h\partial_{\nu_A}\overline{u})dS \\
&= (l+m)I_A + \left(-\frac{a_1}{b}m - \frac{l}{b}n\right) I_B \\
&+ \int_{C'} \left(\left(-(x+m)\frac{f'(x)}{\sqrt{1+f'(x)^2}} + (f(x)+n)\frac{1}{\sqrt{1+f'(x)^2}}\right) h\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS \\
&= 2
\end{aligned}$$

First, as m and n are independent parameters, we can set $m = n = 0$, which yields

$$lI_A + \int_{C'} \left(\left(\frac{-xf'(x)+f(x)}{\sqrt{1+f'(x)^2}}\right) h\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS = 2 \tag{2.1}$$

Additionally, we can differentiate with respect to m:

$$I_A - \frac{a_1}{b}I_B - \int_{C'} \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS = 0 \tag{2.2}$$

and with respect to n:

$$-\frac{l}{b}I_B + \int_{C'} \left(\left(\frac{1}{\sqrt{1+f'(x)^2}}\right) h\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS = 0. \tag{2.3}$$

In equation (2.1), observe that

$$\begin{aligned}
|-xf'(x)+f(x)| &= \left|-x\left(\frac{a_2}{l} + g'(x)\right) + \frac{a_2}{l}x + g(x)\right| \\
&= |g(x) - xg'(x)| \\
&\le |g(x)| + |xg'(x)| \\
&\le \epsilon + l\epsilon \\
&= (l+1)\epsilon
\end{aligned}$$

as $x \leq l$. Plugging this in equation (2.1) yields

$$\begin{aligned}
2 &= lI_A + \int_{C'} \left(\left(\frac{-xf'(x) + f(x)}{\sqrt{1 + f'(x)^2}} \right) h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
&\leq lI_A + \int_{C'} \left(\left(\frac{|-xf'(x) + f(x)|}{\sqrt{1 + f'(x)^2}} \right) h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
&\leq lI_A + \int_{C'} \left(\left(\frac{(l+1)\epsilon}{\sqrt{1 + f'(x)^2}} \right) h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
&= lI_A + \int_{C'} \frac{(l+1)\epsilon}{\sqrt{1 + f'(x)^2}} |h\partial_\nu|^2 dS \\
&\leq lI_A + \int_{C'} (l+1)\epsilon |h\partial_\nu|^2 dS \\
&= lI_A + \beta\epsilon
\end{aligned}$$

where $\beta = \int_{C'} (l+1)|h\partial_\nu|^2 dS = (l+1)I_{C'}$. In the next section, we will show that β is finite and bounded by a number that is independent of ϵ.

Thus, we find the lower bound of I_A to be

$$\frac{2}{l} - \frac{\beta\epsilon}{l}$$

The upper bound of I_A can be found in a similar way to get

$$I_A \leq \frac{2}{l} + \frac{\beta\epsilon}{l}.$$

Hence, we find the range of I_A to be

$$\frac{2}{l} - \frac{\beta\epsilon}{l} \leq I_A \leq \frac{2}{l} + \frac{\beta\epsilon}{l}.$$

Now, comparing to the original triangle T, we have

$$\frac{2}{l} = \frac{a}{al/2} = \frac{a}{\text{Area}(T)},$$

and the perturbation g changes the area by a factor controlled by ϵ:

$$\text{Area}(D) = \text{Area}(T) + \mathcal{O}(\epsilon).$$

Hence

$$\frac{2}{l} = \frac{a}{\text{Area}(D)} + \mathcal{O}(\epsilon).$$

In other words, we find

$$I_A = \frac{a}{Area(D)} + \mathcal{O}(\epsilon).$$

Note that when $\epsilon = 0$, the domain D would be a triangle, and $I_A = \frac{a}{Area(D)}$, so this is consistent with the results in [Chr17].

Next, we substitute $f'(x) = \frac{a_2}{l} + g'(x)$ in equation (2.2) and use equation (2.3) to get

$$\begin{aligned}
I_A - \frac{a_1}{b} I_B - \int_{C'} \left(\frac{f'(x)}{\sqrt{1 + (f'(x))^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
= I_A - \frac{a_1}{b} I_B - \int_{C'} \left(\frac{\frac{a_2}{l} + g'(x)}{\sqrt{1 + (f'(x))^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
= I_A - \frac{a_1}{b} I_B - \frac{l}{b} \frac{a_2}{l} I_B - \int_{C'} \left(\frac{g'(x)}{\sqrt{1 + (f'(x))^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
= I_A - \frac{a}{b} I_B - \int_{C'} \left(\frac{g'(x)}{\sqrt{1 + f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
= 0
\end{aligned}$$

and thus

$$I_B = \frac{b}{a} I_A - \frac{b}{a} \int_{C'} \left(\frac{g'(x)}{\sqrt{1 + f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS$$

As we assume that $|g'(x)|$ is bounded by ϵ, we can find the upper and lower bound of I_B as we did for I_A:

$$\begin{aligned}
I_B &= \frac{b}{a} I_A - \frac{b}{a} \int_{C'} \left(\frac{g'(x)}{\sqrt{1 + (f'(x))^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS \\
&= \frac{b}{a} I_A + \mathcal{O}(\epsilon) \\
&= \left(\frac{b}{a} \right) \frac{a}{\text{Area}(D)} + \mathcal{O}(\epsilon) \\
&= \frac{b}{\text{Area}(D)} + \mathcal{O}(\epsilon).
\end{aligned}$$

Finally, we plug in the range of I_B to equation (2.3) and find

$$\int_{C'} \left(\frac{1}{\sqrt{1 + (f'(x))^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS = \frac{l}{b} I_B \tag{2.4}$$

$$= \frac{l}{\text{Area}(D)} + \mathcal{O}(\epsilon). \tag{2.5}$$

In order to find the range of $\frac{1}{\sqrt{1+f'(x)^2}}$, we use our assumption that $|g'(x)|$ is bounded by ϵ and get

$$\begin{aligned}
(f'(x))^2 &= \left(\frac{a_2}{l} + g'(x) \right)^2 \\
&= \left(\frac{a_2}{l} \right)^2 + \frac{2a_2}{l} g'(x) + g'(x)^2 \\
&= \left(\frac{a_2}{l} \right)^2 + \alpha
\end{aligned}$$

where α is a function of $g'(x)$, $\alpha = \mathcal{O}(\epsilon)$ for ϵ small.

Next, we have

$$\begin{aligned}\frac{1}{\sqrt{1+f'(x)^2}} &= \frac{1}{\sqrt{1+(\frac{a_2}{l})^2+\alpha}}\\ &= \frac{1}{\sqrt{(1+(\frac{a_2}{l})^2)}}\frac{1}{\sqrt{(1+\frac{\alpha}{1+(\frac{a_2}{l})^2})}}\\ &= \frac{l}{c}\frac{1}{\sqrt{1+\frac{l}{c}\alpha}}\\ &= \frac{l}{c}+\mathcal{O}(\epsilon),\end{aligned}$$

where $c=(a_2^2+l^2)^{1/2}$ is the length of side C before deforming it to side C'.

Plugging this result back to equation (2.5) yields

$$\begin{aligned}\frac{2}{a}+\mathcal{O}(\epsilon) &= \int_{C'}\left(\frac{1}{\sqrt{1+f'(x)^2}}h\partial_{\nu_{C'}}u\right)(h\partial_{\nu_{C'}}\overline{u})dS\\ &= \frac{l}{c}(1+\mathcal{O}(\epsilon))\int_{C'}|h\partial_{\nu_{C'}}u|^2dS.\end{aligned} \tag{2.6}$$

We now use again that

$$\text{Area}(D)=\text{Area}(T)+\mathcal{O}(\epsilon)=\frac{al}{2}+\mathcal{O}(\epsilon)$$

where T is the original triangle, and that the length of C' is $l(C')=c+\mathcal{O}(\epsilon)$. Therefore, we have from (2.6)

$$\begin{aligned}\int_{C'}|h\partial_{\nu_{C'}}u|^2dS &= \frac{c}{l}\frac{2}{a}+\mathcal{O}(\epsilon)\\ &= \frac{c}{al/2}+\mathcal{O}(\epsilon)\\ &= \frac{l(C')}{\text{Area}(D)}+\mathcal{O}(\epsilon).\end{aligned}$$

This proves the theorem in the case of an acute or right triangle.

2.3. **Proof of Theorem 1 for obtuse triangles.** The proof of obtuse triangles is nearly the same, with a few changes in the signs. The set up for obtuse triangles is shown in Figure 2.

We can parametrize B and C' with respect to x:

$$B=\{(x,y)\in\mathbb{R}^2: f(x)=\frac{a_1}{l}x, 0\leq x\leq l\}$$

and

$$C'=\{(x,y)\in\mathbb{R}^2: f(x)=\frac{a_2+a_1}{l}x+g(x), 0\leq x\leq l\}$$

Doing similar computations as before we find

$$\begin{aligned}h\partial_y u &= -\frac{l}{b}h\partial_{\nu_B}u\\ h\partial_x u &= \frac{a_1}{b}h\partial_{\nu_B}u\end{aligned}$$

along B, and

$$h\partial_y u = \frac{1}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u$$
$$h\partial_x u = -\frac{f'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u$$

along C'.

Following the commutator computation as in the acute case, and plug in the equation of side B and C', we have

$$2 = (l+m)I_A + \left(\frac{a_1}{b}m - \frac{l}{b}n\right) I_B$$
$$+ \int_{C'} \left(\left(-(x+m)\frac{f'(x)}{\sqrt{1+f'(x)^2}} + (f(x)+n)\frac{1}{\sqrt{1+f'(x)^2}} \right) h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS$$

Differentiating with respect to m and n we get

$$I_A + \frac{a_1}{b} I_B - \int_{C'} \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS = 0 \tag{2.7}$$

and

$$-\frac{l}{b} I_B + \int_{C'} \left(\left(\frac{1}{\sqrt{1+f'(x)^2}} \right) h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS = 0. \tag{2.8}$$

Again, if we set $m = n = 0$, we find the range of I_A to be

$$I_A = \frac{2}{l} + \mathcal{O}(\epsilon) = \frac{a_2}{\text{Area}(D)} + \mathcal{O}(\epsilon).$$

Next, plugging in $f'(x)$ in equation (2.7) and using equation (2.8), we have

$$I_A + \frac{a_1}{b} I_B - \int_{C'} \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS$$
$$= I_A + \frac{a_1}{b} I_B - \int_{C'} \left(\frac{\frac{a_1+a_2}{l} + g'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS$$
$$= I_A + \frac{a_1}{b} I_B - \frac{l}{b}\frac{a_1+a_2}{l} I_B - \int_{C'} \left(\frac{g'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u \right) (h\partial_{\nu_{C'}} \overline{u}) dS$$
$$= I_A - \frac{a_2}{b} I_B - \int_{C'} \left(\frac{g'(x)}{\sqrt{1+f'(x)^2}} h\partial_{\nu_{C'}} u)(h\partial_{\nu_{C'}} \overline{u} \right) dS$$
$$= 0$$

which is the same equation we have for acute triangles.

Therefore, using the range of I_A and the same estimates as in the acute case, we find the range of I_B to be

$$I_B = \frac{2b}{a_2 l} + \mathcal{O}(\epsilon) = \frac{b}{\text{Area}(D)} + \mathcal{O}(\epsilon).$$

Finally, using equation (2.8) and following the computation above, the range of $I_{C'}$ is the same as that of acute triangles:

$$I_{C'} = \frac{2c}{a_2 l} + \mathcal{O}(\epsilon) = \frac{l(C')}{\text{Area}(D)} + \mathcal{O}(\epsilon).$$

3. Proof of Theorem 2

We now proceed with the proof of Theorem 2. It naturally is very similar to that of Theorem 1 so we just point out some of the main differences.

Proof. With the same vector field $X = (x+m)\partial_x + (y+n)\partial_y$, the calculation of the commutator alone tells us that

$$[-h^2 \triangle - 1, X] = -2h^2 \triangle = 2(-h^2 \triangle + w(x,y)) - 2w(x,y)$$

and

$$\begin{aligned}\int_T ([-h^2 \triangle - 1, X]u)\overline{u}dV &= 2\int_T ((-h^2 \triangle + w(x,y))u)\overline{u}dV - 2\int_T (w(x,y)u)\overline{u}dV \\ &= 2 - 2\int_T (w(x,y)u)\overline{u}dV\end{aligned}$$

Because $|w(x,y)| < \epsilon$, we have

$$\begin{aligned}\left|\int_T ([-h^2 \triangle - 1, X]u)\overline{u}dV\right| &= \left|2 - 2\int_T (w(x,y)u)\overline{u}dV\right| \\ &\geq 2 - 2\epsilon \int_T |u|^2 dV \\ &= 2 - 2\epsilon\end{aligned}$$

and

$$\left|\int_T ([-h^2 \triangle - 1, X]u)\overline{u}dV\right| \leq 2 + 2\epsilon.$$

On the other hand, we have

$$\begin{aligned}&\int_T ([-h^2 \triangle - 1, X]u)\overline{u}dV \\ &= \int_T ((-h^2 \triangle - 1)Xu)\overline{u}dV - \int_T (X(-h^2 \triangle - 1)u)\overline{u}dV \\ &= \int_T ((-h^2 \triangle - 1)Xu)\overline{u}dV + \int_T (X(w(x,y)u)\overline{u}dV\end{aligned} \tag{3.1}$$

Integrating by parts, we have

$$\begin{aligned}\int_T ((-h^2 \triangle - 1)Xu)\overline{u}dV &= \int_T (Xu)(-h^2 \triangle - 1)\overline{u}dV \\ &\quad - \int_{\partial T} (h\partial_\nu hXu)\overline{u}dS + \int_{\partial T} (hXu)(h\partial_\nu \overline{u})dS \\ &= -\int_T (Xu)w(x,y)\overline{u}dV + \int_{\partial T} (hXu)(h\partial_\nu \overline{u})dS.\end{aligned} \tag{3.2}$$

The last term in (3.1) is computed:

$$\begin{aligned}\int_T (X(w(x,y)u)\overline{u}dV &= \int_T ((Xw(x,y))u) + (w(x,y)Xu))\overline{u}dV \\ &= \int_T ((Xw(x,y))u)\overline{u}dV + \int_T (w(x,y)Xu)\overline{u}dV.\end{aligned}$$

Combining this with (3.1) and (3.2), we have

$$\begin{aligned}&\int_T ([-h^2\triangle - 1, X]u)\overline{u}dV \\ &= \int_T ((-h^2\triangle - 1)Xu)\overline{u}dV + \int_T (X(w(x,y)u)\overline{u}dV \\ &= \int_{\partial T} (hXu)(h\partial_\nu \overline{u})dS + \int_T ((Xw(x,y))u)\overline{u}dV.\end{aligned}$$

With the condition that $|\nabla w(x,y|) \leq \epsilon$, we get

$$\begin{aligned}&\left|\int_T ((Xw(x,y))u)\overline{u}dV\right| \\ &= \left|\int_T (((x+m)\partial_x w(x,y) + (y+n)\partial_y w)x,y))u)\overline{u}dV\right| \\ &= \mathcal{O}(\epsilon) + m\mathcal{O}(\epsilon) + n\mathcal{O}(\epsilon).\end{aligned}$$

Combining the results together, we have

$$\begin{aligned}2 + \mathcal{O}(\epsilon) + m\mathcal{O}(\epsilon) + n\mathcal{O}(\epsilon) &= \int_T ([-h^2\triangle - 1, X]u)\overline{u}dV \\ &= \int_{\partial T} (hXu)(h\partial_\nu \overline{u})dS.\end{aligned}$$

The rest of the proof proceeds exactly as the proof of Theorem 1. □

4. Proof of Lemma 2.1

Proof. To prove the Lemma, first we consider the vector field

$$X = y\partial_y$$

and the usual computation yields

$$[-h^2\triangle - 1, y\partial_y] = -2h^2\partial_y^2$$

Then the integration yields

$$\int_D ([-h^2\triangle - 1, y\partial_y]u)\overline{u}dV = -2\int_D (h^2\partial_y^2 u)\overline{u}dV$$

Moreover, observe that

$$\begin{aligned}-2\int_D (h^2\partial_y^2 u)\overline{u}dV &= 2\int_D |h\partial_y u|^2 dV \\ &\leq 2\int_D (|h\partial_y u|^2 + |h\partial_x u|^2)dV \\ &= 2\int_D |u|^2 dV \\ &= 2\end{aligned}$$

On the other hand, if we integrate by parts and using the boundary conditions, we have

$$\int_D([-h^2\triangle-1, y\partial_y]u)\overline{u}dV = \int_D((-h^2\triangle-1)y\partial_y u)\overline{u}dV$$
$$= \int_{\partial D}(yh\partial_y u)(h\partial_\nu\overline{u})dS$$

Hence, together we have

$$-2\int_D(h^2\partial_y^2 u)\overline{u}dV = \int_{\partial D}(yh\partial_y u)(h\partial_\nu\overline{u})dS \le 2$$

Since A is vertical, we have

$$\partial_y u = 0$$

on A,

$$\partial_y u = -\frac{l}{b}\partial_\nu$$

on B (in both the acute and obtuse cases), and

$$\begin{aligned}\partial_y u &= \frac{1}{\sqrt{1+(f'(x))^2}}\partial_\nu u \\ &= \gamma^{-1}\partial_\nu u\end{aligned}$$

on C', where $\gamma = \sqrt{1+(f'(x))^2}$ is the arclength element. Substituting $\partial_y u$, we have

$$\begin{aligned}&\int_{\partial D}(yh\partial_y u)(h\partial_\nu\overline{u})dS \\ &= \int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS + \int_B\left(-\frac{l}{b}\right)\left(\mp\frac{a_2}{l}x\right)|h\partial_\nu u|^2 dS \\ &= \int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS \pm \frac{a_2}{b}\int_B x|h\partial_\nu u|^2 dS \\ &\le 2,\end{aligned} \tag{4.1}$$

where the $\pm$ sign corresponds to the acute/obtuse cases.

While this is close to what we intend to prove, we should be careful because $f(x)$ approaches zero as x goes to zero. There is also a potential problem in the obtuse case because of the sign change on the B integral.

Acute case: Observe that in (4.1) the function $a_2x/b \ge 0$, so in the acute case, we have

$$\begin{aligned}&\left|\int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS\right| \\ &= \int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS \\ &\le \int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS + \frac{a_2}{b}\int_B x|h\partial_\nu u|^2 dS \\ &\le 2\end{aligned}$$

so to show that $\int_{C'}|h\partial_\nu u|^2 dS$ is bounded we only need to estimate the integral for x near 0.

Fix $\delta > 0$ such that $\epsilon < \frac{a\delta}{2b}$ and $\delta \gg \epsilon$. Using our restrictions on $f(x)$, we can find the lower bound of $f(x)$ when $x \geq \delta$:

$$\begin{aligned} f(x) &= \frac{a}{b}x + g(x) \\ &\geq \frac{a}{b}x - |g(x)| \\ &\geq \frac{a}{b}x - \epsilon \\ &\geq \frac{a\delta}{b} - \epsilon \\ &\geq \frac{a\delta}{2b}. \end{aligned}$$

For $\delta \leq x \leq l$, we have an upper bound for γ^{-1}:

$$\begin{aligned} \gamma &= \sqrt{1 + (f'(x))^2} \\ &= \sqrt{1 + \Big(\frac{a}{b} + g'(x)\Big)^2} \\ &\leq \sqrt{1 + \Big(\frac{a}{b} + \epsilon\Big)^2} \\ &=: \gamma_0. \end{aligned}$$

We observe that then

$$\begin{aligned} \gamma^{-1} \geq \gamma_0^{-1} &= \frac{1}{\sqrt{1 + \frac{a^2}{b^2}}} + \mathcal{O}(\epsilon) \\ &\geq \frac{1}{2\sqrt{1 + \frac{a^2}{b^2}}} \end{aligned}$$

for $\epsilon > 0$ sufficiently small.

Hence, substituting $f(x)$ and γ^{-1} with their lower bounds, we have

$$\begin{aligned} 2 &\geq \int_{C'} f(x)\gamma^{-1}|h\partial_\nu u|^2 dS \\ &\geq \gamma_0^{-1}\frac{a\delta}{2b}\int_{C'\cap\{x\geq\delta\}} |h\partial_\nu u|^2 dS \\ &\geq \frac{a\delta}{4b\sqrt{1 + \frac{a^2}{b^2}}}\int_{C'\cap\{x\geq\delta\}} |h\partial_\nu u|^2 dS. \end{aligned} \tag{4.2}$$

That means that (4.2) implies

$$\int_{C'\cap\{x\geq\delta\}} |h\partial_\nu u|^2 dS = \mathcal{O}_\delta(1)$$

independent of ϵ provided ϵ is sufficiently small.

For $x < \delta$, consider another function $\psi(x)$, which has value one on $x \leq \delta$ and monotonically decreases to zero for $x \geq 2\delta$. Such a function ψ is depicted in Figure 3.

Then computing the commutator we have

$$[-h^2\triangle - 1, \psi\partial_x] = -2\psi' h^2\partial_x^2 - h\psi'' h\partial_x$$

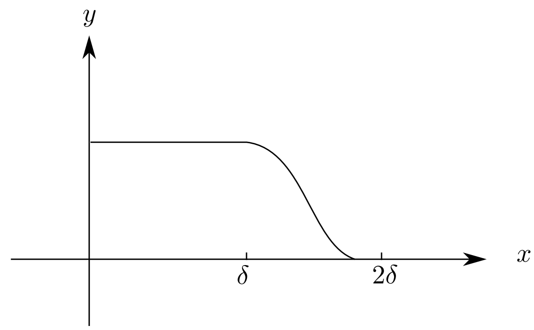

Figure 3. The function ψ

Thus we have

$$\begin{aligned}\left|\int_D ([-h^2\triangle - 1, \psi\partial_x]u)\overline{u}dV\right| &= \left|\int_D (2\psi' h^2\partial_x^2 u)\overline{u} + (h\psi'' h\partial_x u)\overline{u}dV\right| \\ &= \left|\int_D (-2h\partial_x u h\partial_x \psi'\overline{u})dV + \int_D (h\psi'' h\partial_x u)\overline{u}dV\right| \\ &= \left|\int_D (-h\partial_x u h\psi''\overline{u} - 2\psi'|h\partial_x u|^2)dV\right| \\ &\leq \sup(|2\psi'|, |\psi''|)\int_D (|h\partial_x u||hu| + |h\partial_x u|^2)dV \\ &= \mathcal{O}_\delta(1).\end{aligned}$$

Here the implicit constant in the $\mathcal{O}_\delta(1)$ depends on our fixed δ, but not on $\epsilon \ll \delta$.

On the other hand, if we integrate by parts, we have

$$\int_D ([-h^2\triangle - 1, \psi\partial_x]u)\overline{u}dV = \int_{\partial D} \psi h\partial_x u h\partial_\nu \overline{u}dS = \mathcal{O}_\delta(1).$$

Since

$$\partial_x u = -\frac{f'}{\gamma}\partial_\nu u$$

on C' and we have already computed

$$\begin{aligned}\left|\frac{f'}{\gamma}\right| &= \left|\frac{\frac{a_2}{l} + g'}{\gamma}\right| \\ &\geq \frac{a_2}{4l\sqrt{1+\frac{a^2}{b^2}}}\end{aligned}$$

for ϵ sufficiently small, we have

$$\begin{aligned}\int_{C'\cap\{0\leq x\leq\delta\}} |h\partial_\nu u|^2 dS &\leq \int_{C'} \psi(x)|h\partial_\nu u|^2 dS \\ &\leq \frac{4l\sqrt{1+\frac{a^2}{b^2}}}{a_2}\int_{C'} \psi\frac{f'}{\gamma}|h\partial_\nu u|^2 dS \\ &= \mathcal{O}_\delta(1).\end{aligned}$$

Combining with (4.2), we have

$$\int_{C'} |h\partial_\nu u|^2 dS = \mathcal{O}(1),$$

which proves the Lemma in this cases.

Case of obtuse triangle: In this case we have to be slightly more careful. Consider the vector field

$$X = \left(y - \frac{a_2}{l}x\right)\partial_y.$$

We have $[-h^2 \triangle - 1, X] = -2h^2\partial_y^2 + 2\frac{a_2}{l}h\partial_x h\partial_y$. The interior estimates are similar, so that

$$\int_D ([-h^2 \triangle - 1, X]u)\overline{u}dV = \mathcal{O}(1).$$

Then the vector field X vanishes when $y = \frac{a_2}{l}x$. Further, since X is tangential on side A, fixing a $\delta \gg \epsilon$, the same argument as in the acute case gives

$$\int_{C' \cap \{\delta \le x \le l\}} |h\partial_\nu u|^2 dS = \mathcal{O}(1).$$

For the set $\{0 \le x \le \delta\}$, we use the vector field $Y = \psi(x)\left(\partial_x + \frac{a_2}{l}\partial_y\right)$. Then $Y = 0$ on A since $\psi = 0$ there, and Y is tangential to B, so $Yu = 0$ on B.

On C', we have

$$hYu = \psi(x)\left(-\frac{f'}{\gamma} + \frac{a_1}{l\gamma}\right)h\partial_\nu u.$$

Since in the obtuse case we have $f' = (a_1 + a_1)/l + g'$,

$$\frac{f' - \frac{a_1}{l}}{\gamma} = \frac{a_2}{l\gamma} + \mathcal{O}(\epsilon) \ge \frac{a_2}{2l\gamma}$$

independent of ϵ sufficiently small. Using the previously established estimates on γ, the rest of the proof follows exactly as in the acute case.

□

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