A ONE POINT NON-CONCENTRATION ESTIMATE FOR LAPLACE EIGENFUNCTIONS ON POLYGONS

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Abstract. In this paper we consider eigenfunctions of the Laplacian on a planar domain with polygonal boundary with Dirichlet, Neumann, or mixed boundary conditions. The main result is a quantitative estimate on the L^2 mass of eigenfunctions near a point in terms of the distance to the nearest non-adjacent boundary face. In particular, eigenfunctions cannot concentrate completely at any one single point. The technique of proof is to use the commutator ideas from the recent work of the author [Chr17, Chr18] on triangles and simplices.

1. Introduction

In this paper, we study the distribution of interior L^2 mass of Laplace eigenfunctions on polygonal domains with Dirichlet, Neumann, or mixed boundary conditions. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set. Eigenfunctions on Ω are used to model, for example, the fundamental modes of vibration for a drum with shape Ω, as well as other physical phenomena. Eigenfunctions are highly sensitive to the geometry of the boundary of Ω and the boundary conditions imposed on $\partial\Omega$. This is part of the "classical-quantum correspondence". The classical problem in a planar domain is to consider the trajectories of a billiard ball on a table shaped like Ω. An ideal billiard ball will follow a straight line until it meets the boundary, at which point it will reflect according to Snell's law (angle in equals angle out). If the boundary of Ω is sufficiently smooth, then one can describe the billiard trajectories as a map on the closed co-ball bundle $\overline{B^*\partial\Omega}$, by specifying the point of impact on the boundary and the incoming direction. Consider now a wave on Ω. Waves tend to travel in packets along straight lines in planar domains as well, and reflect off boundary walls according to Snell's law. But wave packets cannot be completely localized to a single billiard ball trajectory, so they can do crazy things when they reflect off a wall, and the curvature and regularity of the boundary of Ω at the reflection point can cause wave packets to focus, de-focus, disperse, diffract, glance, and many other possibilities.

If $\partial\Omega$ is a closed polygonal path, then at each corner the boundary has only Lipschitz regularity, while away from the corners, the boundary is affine, so $\mathcal{C}^\infty$. Imagining a billiard ball on a polygonal domain Ω, one begins to see subtleties even in the classical problem. How does one specify how a billiard ball reflects when it heads into a corner?

By separation of variables, the study of solutions to the wave equation on Ω can be reduced to studying eigenfunctions. In this paper, we consider the following eigenfunction problem. Let $\{u_j\}$ be a sequence of functions satisfying

$$\begin{cases} -\Delta u_j = \lambda_j^2 u_j, \text{ in } \Omega, \\ Bu_j = 0 \text{ on } \partial\Omega, \\ \|u_j\|_{L^2(\Omega)} = 1. \end{cases} \tag{1.1}$$

Here B is a boundary operator, $Bu_j = u_j$ or $Bu_j = \partial_\nu u_j$ on each affine segment of $\partial\Omega$ (see Subsection 2.1 for a rigorous definition). It is classical that this problem has a countably infinite number of solutions, from which one obtains an orthonormal basis for L^2. With this sequence of eigenfunctions comes a sequence of eigenvalues λ_j^2, with $\lambda_j \to \infty$. The size of the λ_j^{-1}s corresponds to the wavelength, so λ_j is proportional to the frequency of oscillation of associated waves. Hence the limit as $\lambda_j \to \infty$ describes "high-frequency" behaviour of eigenfunctions.

The goal of this paper is to study how eigenfunctions can concentrate (or more precisely, *not* concentrate) in the limit $\lambda_j \to \infty$. Non-concentration results have a huge history, which we briefly discuss below in Subsection 1.1. Many of the techniques used in these previous works make use of *microlocal analysis*, which is the study of waves in both space and frequency, using knowledge about the associated classical problem. This means that to study eigenfunctions using these tools, a certain amount of regularity of the boundary is necessary. This is so that the classical billiard map is well-defined, and the billiard flow is sufficiently smooth that quantum observables can be constructed roughly as functions of the billiard flow.

Of course in the case of polygonal domains, the billiard flow is generally not smooth, and many techniques from microlocal analysis break down. In the present work, we bypass the use of microlocal analysis in favor of commutator methods. This has the virtue of being very robust and requiring very few assumptions on the regularity of the boundary. However, it has the defect of losing any information about the *frequency* localization of eigenfunctions.

The main result in this paper is a quantitative estimate on the L^2 mass of a sequence of eigenfunctions in a neighbourhood of a single point. A novelty of this work is that we also obtain a quantitative estimate in a neighbourhood of a boundary point as well.

Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be a connected, bounded, and open set with polygonal boundary, and let $\{u_j\}$ be a sequence of eigenfunctions on Ω satisfying* (1.1). *Let $p_0 \in \overline{\Omega}$ be any point (including on the boundary), and let $d = dist(p_0, F') > 0$, where $F' \subset \partial\Omega$ is the nearest boundary face not adjacent to p_0. Then for each $0 < \alpha < 1$,*

$$\limsup_{\lambda_j \to \infty} \|u\|^2_{L^2(D(p_0, \alpha d))} \leqslant \frac{1}{2-\alpha}. \tag{1.2}$$

Remark 1.1. Put another way, Theorem 1 says that any semiclassical defect measure associated to the sequence $\{u_j\}$ cannot be supported at only a single point, including boundary points.

Remark 1.2. The lower bound on the mass outside of a neighbourhood of p_0 independent of λ_j is very strong. In general, given any open, bounded subset $U \subset \Omega$, Carlemann estimates give an exponential lower bound

$$\|u_j\|_{L^2(U)} \geqslant ce^{-c\lambda_j}$$

for some $c > 0$. The tradeoff is that for our result U is a large subset; the complement of a neighbourhood of a single point. In particular, our theorem does not rule out the possibility of glancing modes, which can concentrate very strongly in a neighbourhood of size $\lambda_j^{-2/3}$ of the boundary due to Airy asymptotics of glancing modes (see [AS64, Mel76] for a discussion of Airy asymptotics and glancing, and [CHT15] for a discussion of optimality).

Remark 1.3. In this paper, we consider only *classical* polygons, in the sense that the boundary is a closed, piecewise affine curve with finitely many corners. Each corner makes an interior angle θ satisfying $0 < \theta < 2\pi$.

If $p_0 \in \partial\Omega$, then p_0 is either on the interior of an open face or it is a corner. If p_0 is a corner making an angle $0 < \theta < \pi$, we call it a *convex* corner, and if the angle is $\pi < \theta < 2\pi$, we call it a *concave* corner. In all cases, we prove a *quantitative* estimate in terms of distance to the nearest non-adjacent side. This is stated concretely in Propositions 3.1 and 4.1.

1.1. **History.** Non-concentration type estimates have an enormous history for several reasons. Research into properties of eigenfunctions is an old subject - for example Fourier series are eigenfunction expansions. The connections to physical phenomena and other areas of mathematics (acoustics, scattering, tunneling, elliptic equations, quantum chaos, number theory, etc.) make qualitative properties of eigenfunctions an important and continuing area of research. Many questions about eigenfunctions are relatively concrete to state, allowing even undergraduate students to at least understand the questions. (And prove new interesting results! The author has several students working on related problems.)

The strongest non-concentration is equidistribution in phase space. If the classical billiard flow is *ergodic* (roughly "chaotic"), then the eigenfunctions are known to equidistribute in phase space (at least along a density one subsequence) [Šni74, Zel87, CdV85]. That means, except possibly for a density zero subsequence, the eigenfunctions concentrate equally everywhere. This is called *quantum ergodicity.*

These possible exceptional subsequences of *non*-quantum ergodic eigenfunctions have also been heavily studied. In the case of joint Hecke-Laplace eigenfunctions Lindenstrauss proved there are no exceptional subsequences, a property called quantum unique ergodicity [Lin06]. In the case of the Bunimovich stadium Hassell proved that as one varies the length of the rectangular part of the stadium, quantum unique ergodicity fails with probability 1 [Has10]. In the works [CdVP94a, CdVP94b, Chr07, Chr11], it is shown that eigenfunctions cannot concentrate too sharply along an unstable periodic geodesic. That gives a restriction on what possible limit measures exist for sequences of eigenfunctions in this case.

On the other hand, as mentioned above, it is possible for eigenfunctions to concentrate very sharply in a λ dependent neighbourhood of a hypersurface, or even near a single point. Further, by measuring L^p norms instead of L^2 norms, even more refined concentration/non-concentration estimates are known [Sog86, Sog88]. It should be noted that these estimates are sharp on the sphere, which clearly plays no role in the present paper!

Yet another measure of concentration/non-concentration is to consider restrictions of eigenfunctions to lower dimensional sets. Although this is not the topic of this paper, it is worthwhile to mention a few results. Burq-Gérard-Tzvetkov [BGT07] give sharp upper bounds on restrictions of eigenfunctions and the author's work with Hassell-Toth [CHT15] gives sharp upper bounds on the Neumann data on hypersurfaces. In the setting of quantum ergodic eigenfunctions, more is known [GL93, HZ04, TZ12, TZ13, CTZ13, DZ13], however again these results require smoothness of the domain and a priori knowledge about ergodicity of the classical flow. The novel feature of the author's work on boundary values on triangles and simplices [Chr17, Chr18] is that it does not require any dynamical systems knowledge, and only relies on integrations by parts. This is the starting point for the present paper.

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2. Preliminaries

Here we record some notation and basic facts needed to complete the quantitative estimates in Propositions 3.1 and 4.1 below.

2.1. **Notation.**

In this subsection, we define more precisely what is meant by polygonal boundary and the boundary operator B used in Theorem 1.

Let $\Gamma = \cup_{j=1}^N \Gamma_j \subset \mathbb{R}^2$ be a union of disjoint closed simple polygonal curves, and let $\Omega \subset \mathbb{R}^2$ be the bounded open domain enclosed by Γ. In order to have consistent normalization for the eigenfunctions, let us assume that Ω is connected. For each $1 \leqslant j \leqslant N$, Γ_j is a union of a finite number of linear segments $\Gamma_j = \cup_{k_j=1}^{M_j} \Gamma_{k_j}$. On each segment, let ν_{k_j} denote the outward (with respect to Ω) unit normal vector, and let B_{k_j} be a homogeneous boundary operator:

$$B_{k_j} u = \begin{cases} u|_{\Gamma_{k_j}}, \text{ for Dirichlet boundary conditions, or} \\ \partial_{\nu_{k_j}} u|_{\Gamma_{k_j}} \text{ for Neumann boundary conditions.} \end{cases}$$

Let $B = \sum_{j=1}^N \sum_{k_j=1}^{M_j} B_{k_j}$ denote the total boundary operator.

Let us drop the subscript j notation on λ_j and rescale $h = \lambda^{-1}$ so that the solutions of (1.1) satisfy the semiclassical eigenfunction problem

$$\begin{cases} -h^2 \Delta u = u \text{ in } \Omega, \\ Bu = 0 \text{ on } \partial\Omega, \\ \|u\|_{L^2(\Omega)} = 1. \end{cases} \tag{2.1}$$

We are then interested in asymptotics as $h \to 0$, and when we write u, we implicitly mean $u = u(h)$ is a sequence of eigenfunctions depending on h.

If $p_0 \in \overline{\Omega}$, then p_0 is either in the interior of Ω, a boundary point on the interior of an edge, a boundary point at a convex corner, or a boundary point at a concave corner. These cases are discussed in Sections 3-4.

2.2. **Regularity of eigenfunctions on polygons.** Before continuing, we remark briefly about regularity of eigenfunctions, which will allow us to perform the necessary integrations by parts. The book of Grisvard [Gri11] has a very detailed account of regularity for elliptic equations on non-smooth domains. In [Chr18], the author studied eigenfunctions on simplices, which are convex and [Gri11] contains the necessary results for that paper. In the present paper, in general a polygon is not convex (and indeed we study eigenfunctions near concave corners), however even more detailed information is available in [Gri11] for polygons. In particular, polygons are Lipschitz domains, and [Gri11, Theorem 1.4.4.6] gives continuity of the derivative from H^m to H^{m-1}, as long as $m \neq 1/2$. Theorem 4.3.1.4 in [Gri11] gives elliptic regularity estimates on polygons, so that we can conclude that eigenfunctions are in H^m for each $m \geqslant 0$. Then any derivatives of eigenfunctions are in H^1, so [Gri11, Theorem 1.5.3.1] shows we can integrate by parts using Green's formula. We finally observe that, by the same results, multiplying an eigenfunction by a smooth bounded function does not decrease its regularity.

2.3. **The radial vector field.** Here we recall some identities involving the radial vector field $r\partial_r$ (in polar coordinates). The reason for this is two-fold. First, commuting with the Laplacian reproduces the Laplacian. The second is that, fixing a point $p_0 \in \partial\Omega$ and translating so that $p_0 = 0$, the vector field $r\partial_r$ is tangential to any segments emanating from 0. This is especially convenient when applied to functions with Dirichlet boundary conditions $u|_{\partial\Omega} = 0$, as then locally $r\partial_r u = 0$ on the boundary since this is a tangential derivative of 0. What is not so obvious is that the radial vector field is also well-behaved when applied to Neumann or mixed boundary conditions. This is discussed in Lemma 4.2 in Section 4 below.

For completeness, let us state the well known commutator and change of variable formulae used in this paper.

Lemma 2.1. *Let $\mathbb{R}^n = (r, \theta)$ be polar coordinates with $0 \leqslant r < \infty$ and $\theta \in \mathbb{S}^{n-1}$. The polar Laplacian is*

$$-\Delta = -\partial_r^2 - \frac{(n-1)}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}},$$

where $-\Delta_{\mathbb{S}^{n-1}}$ is the Laplacian on the unit $n-1$ sphere. Let $X = r\partial_r$. Then

$$[-\Delta, X] = -2\Delta.$$

Now let $\mathbb{R}^n = (x_1, \ldots, x_n)$ be rectangular coordinates with $x_j \in \mathbb{R}$ for each j. The Laplacian in rectangular coordinates is $-\Delta = -\partial_{x_1}^2 - \ldots - \partial_{x_n}^2$, and the radial vector field is $X = x_1\partial_{x_1} + \ldots + x_n\partial_{x_n}$, so that $[-\Delta, X] = -2\Delta$.

The proof is a standard computation and change of variables.

The reason for recording these two coordinate versions is that sometimes it is more convenient to use one or the other. In this paper, for interior points, we are integrating over a disc, so using a vector field which is independent of θ is reasonable. As mentioned above, the radial vector field is tangential to any segment emanating from 0, which is

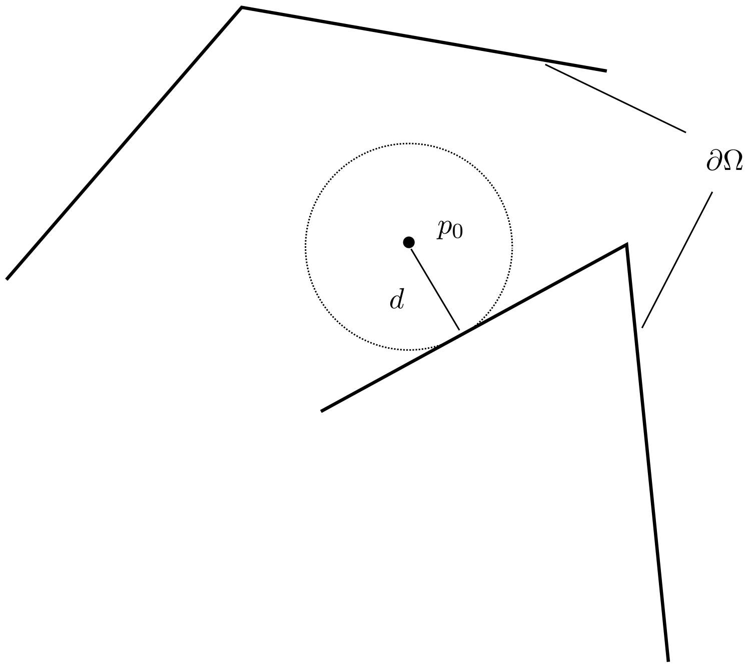

Figure 1. The polygon Ω near p_0 when p_0 lies in the interior of Ω. The distance d is to the closest point on the boundary. Here $\partial\Omega$ is in bold, and the circle of radius d is in dashes. We integrate over a disc of radius αd with $0 < \alpha < 1$.

convenient for Dirichlet boundary conditions on a polygon. But we are also interested in mixed boundary conditions, for which it seems easier to use rectangular coordinates.

3. Proof of Theorem 1 for interior points

In this Section, we study non-concentration near a point on the interior of Ω. It has been known for some time that eigenfunctions cannot concentrate away from the boundary of a polygon [HHM09, MR12] (at least for a subsequence of density one), but we provide a proof in the case of a single point as the estimates are quantitative in this case. In the notation from Section 2, the statement of Theorem 1 is given in the following Proposition.

Proposition 3.1. *Let u be a solution of* (2.1)*, and let $p_0 \in \Omega$. Let $d = dist(p_0, \partial\Omega) > 0$. For each $0 < \alpha < 1$,*

$$\limsup_{h \to 0+} \|u\|^2_{L^2(D(p_0, \alpha d))} \leqslant \frac{1}{2 - \alpha}. \tag{3.1}$$

Proof. We argue by contradiction. Suppose (3.1) is false. Then there exists $\alpha > 0$ and a subsequence $\{h_j\}$ as $h \to 0$ such that

$$\|u(h_j)\|^2_{L^2(D(p_0, \alpha d))} \to R > \frac{1}{2 - \alpha}.$$

Now choose $\epsilon > 0$ satisfying $\epsilon \leqslant \min(\frac{d(1-\alpha)}{4}, \frac{1}{2})$. Shrinking $\epsilon > 0$ further if necessary, we may assume

$$\lim_{j\to\infty} \|u(h_j)\|^2_{L^2(D(p_0,\alpha d))} \geqslant \frac{1}{2-\alpha} + d\epsilon.$$

For ease in exposition, let us immediately drop the subscript and subsequence notation, and just consider a sequence of eigenfunctions u satisfying

$$\|u\|^2_{L^2(D(p_0,\alpha d))} \geqslant \frac{1}{2-\alpha} + d\epsilon + o(1)$$

as $h \to 0$.

Translating Ω, we may assume $p_0 = 0$. We introduce polar coordinates (r, θ) near 0, so that our norm above is over $\{r \leqslant \alpha d\}$. Let $\varphi = \varphi(r)$ be the function φ_1 described in Lemma A.1 with $\delta_1 = \alpha d$, $\delta_2 = d$, and with $\epsilon > 0$ specified above, so that $\varphi(r) \equiv 1$ for $r \leqslant \alpha d + \epsilon^3$, $\varphi(r) \equiv 0$ for $r \geqslant d - \epsilon^3$. Lemma A.1 then guarantees we can choose such a $\varphi(r)$ so that

$$|\varphi'| \leqslant \frac{2}{(1-\alpha)d} + \epsilon.$$

Choose also $\psi(r) \in \mathcal{C}^\infty(\mathbb{R})$ satisfying $\psi \equiv 1$ on $\operatorname{supp}\varphi'$, $0 \leqslant \psi \leqslant 1$, and $\operatorname{supp}\psi(r) \subset [\alpha d, d]$.

Now in polar coordinates, $-h^2\Delta = -h^2\partial_r^2 - hr^{-1}h\partial_r - h^2r^{-2}\partial_\theta^2$, and $[-h^2\Delta, r\partial_r] = -2h^2\Delta$. Hence we have

$$\begin{aligned}\int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV &= -2\int_\Omega \varphi(h^2\Delta u)\bar{u}dV \\ &= 2\int_\Omega \varphi|u|^2 dV \\ &\geqslant 2\left(\frac{1}{2-\alpha} + d\epsilon\right) + o(1), \end{aligned} \tag{3.2}$$

since $\varphi \equiv 1$ on $D(p_0, \alpha d)$ and $\varphi \geqslant 0$ everywhere. On the other hand,

$$\begin{aligned}&\int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV \\ &\quad= \int_\Omega \varphi((-h^2\Delta - 1)r\partial_r u)\bar{u}dV - \int_\Omega \varphi r(\partial_r(-h^2\Delta - 1)u)\bar{u}dV \\ &\quad= \int_\Omega \varphi((-h^2\Delta - 1)r\partial_r u)\bar{u}dV \\ &\quad= \int_\Omega (r\partial_r u)((-h^2\Delta - 1)\varphi\bar{u})dV.\end{aligned}$$

Here we have used the eigenfunction equation (2.1) and that φ has compact support inside Ω so there are no boundary terms when integrating by parts. The last term also vanishes up to commuting with φ, so we have

$$\begin{aligned}
&\int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV \\
&\quad = \int_\Omega (r\partial_r u)([-h^2\Delta, \varphi]\bar{u})dV \\
&\quad = \int_\Omega (r\partial_r u)(-2h\varphi' h\partial_r \bar{u})dV + \mathcal{O}(h) \\
&\quad \leqslant 2\sup|\varphi'| \int \psi r|h\partial_r u|^2 r dr d\theta + \mathcal{O}(h) \\
&\quad \leqslant 2\left(\frac{1}{(1-\alpha)d} + \epsilon\right) d \int \psi |h\partial_r u|^2 r dr d\theta + \mathcal{O}(h).
\end{aligned}$$

To estimate the last term, we use

$$\begin{aligned}
\int \psi|h\partial_r u|^2 r dr d\theta &\leqslant \int \psi(|h\partial_r u|^2 + |r^{-1}h\partial_\theta u|^2) r dr d\theta \\
&= \int \psi(-h^2\Delta u)\bar{u}dV + \mathcal{O}(h) \\
&= \int \psi|u|^2 dV + \mathcal{O}(h) \\
&\leqslant \left(1 - \frac{1}{2-\alpha} - d\epsilon\right) + o(1).
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV \\
&\quad \leqslant 2\left(\frac{1}{(1-\alpha)d} + \epsilon\right) d \int \psi|h\partial_r u|^2 r dr d\theta + o(1) \\
&\quad \leqslant 2d\left(\frac{1}{(1-\alpha)d} + \epsilon\right)\left(1 - \frac{1}{2-\alpha} - d\epsilon\right) + o(1) \\
&\quad \leqslant \frac{2}{1-\alpha} + 2d\epsilon - \frac{2}{(1-\alpha)(2-\alpha)} - \frac{2d\epsilon}{1-\alpha} + o(1) \\
&\quad \leqslant \frac{2}{1-\alpha} - \frac{2}{(1-\alpha)(2-\alpha)} + o(1) \\
&\quad \leqslant \frac{2}{2-\alpha} + o(1). \qquad (3.3)
\end{aligned}$$

Combining (3.2) and (3.3), we have

$$\begin{aligned}
2\left(\frac{1}{2-\alpha} + d\epsilon\right) &\leqslant \int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV \\
&\leqslant \frac{2}{2-\alpha} + o(1),
\end{aligned}$$

which is a contradiction for $h > 0$ sufficiently small.

$\square$

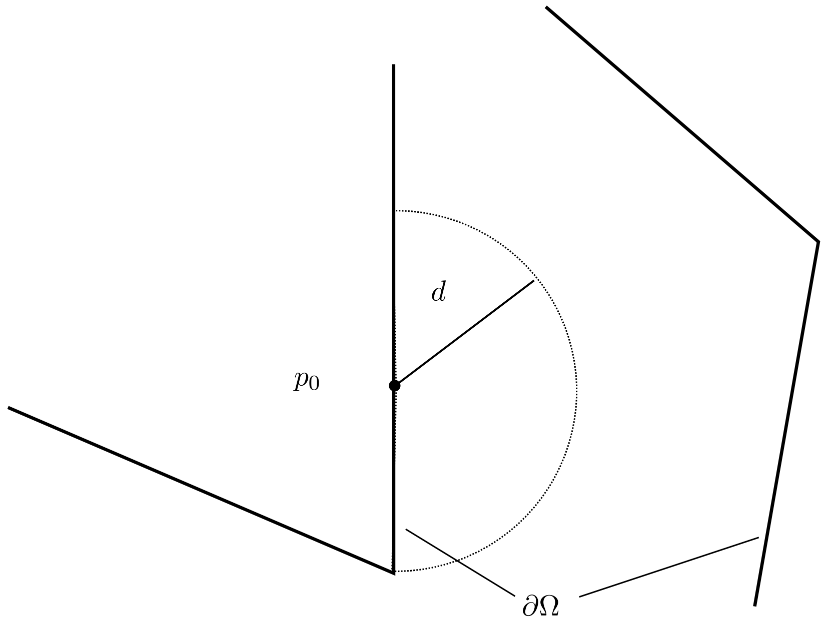

FIGURE 2. The polygon Ω near $p_0 \in \partial\Omega$ when p_0 lies on the interior of a side. The distance d is to the closest vertex or closest other side, whichever is closer. Here $\partial\Omega$ is in bold, and the circle of radius d is in dashes.

4. Proof of Theorem 1 for boundary points

For a boundary point $p_0 \in \partial\Omega$, p_0 either lies in the interior of a flat face, or at a corner. If at a corner, it can either be a convex or concave corner. But it turns out that the proof works more or less the same for all three cases. In fact, the proof is nearly identical to the proof of the interior case. Let θ_0 be the angle of $\partial\Omega$ at p_0, measured from the interior of Ω. If $0 < \theta_0 < \pi$, p_0 lies at a convex corner, and if $\pi < \theta_0 < 2\pi$, p_0 lies at a concave corner. If $\theta_0 = \pi$, then p_0 lies on the interior of a face of $\partial\Omega$. The statement of Theorem 1 for boundary points in the notation of Section 2 is given in the following Proposition.

Proposition 4.1. *Let u be a solution of* (2.1)*, and let $p_0 \in \partial\Omega$. Let $d = dist(p_0, F')$, where F' is the nearest edge to p_0 which is not adjacent to p_0. Then for each $0 < \alpha < 1$,*

$$\limsup_{h\to 0+} \|u\|^2_{L^2(D(p_0,\alpha d)\cap\Omega)} \leqslant \frac{1}{2-\alpha}. \tag{4.1}$$

Before jumping into the proof, we need some knowledge of how mixed boundary conditions interact with the radial vector field.

Lemma 4.2. *Let $p_0 \in F$ for some face F, and let $d = dist(p_0, F')$, where F' is the closest non-adjacent face to p_0. Let u be a solution to* (2.1) *subject to the boundary conditions $Bu = 0$. Then for any $0 < R < d$ and any radial $\varphi \in \mathcal{C}_c^\infty(D(p_0, R))$,*

$$2\int_\Omega \varphi|u|^2 dV = \int_\Omega (Xu)([-h^2\Delta, \varphi]\bar{u})dV.$$

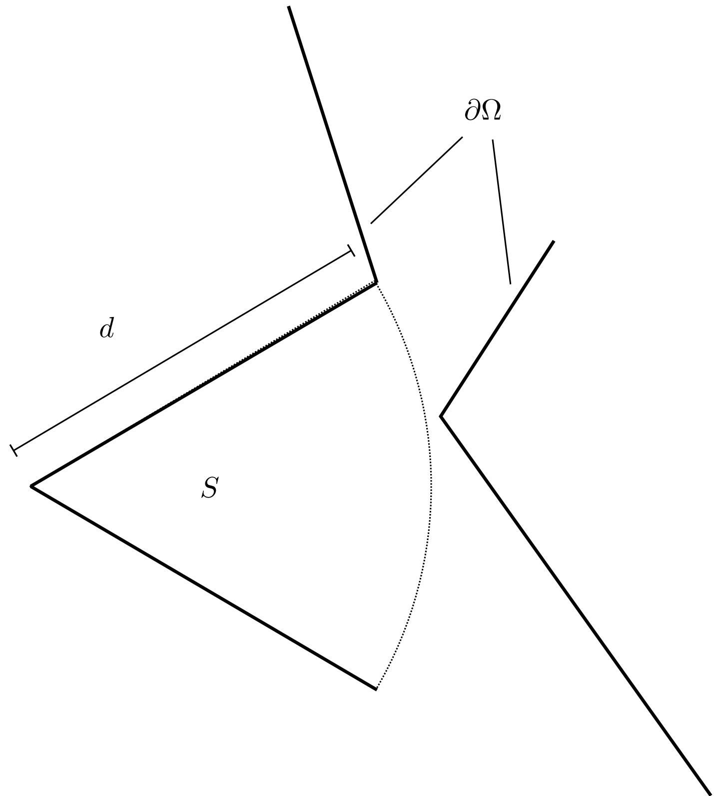

Figure 3. The polygon Ω near $p_0 \in \partial\Omega$ when p_0 lies at a convex corner. The domain Ω is to the right of p_0, so that the interior angle θ_0 satisfies $0 < \theta_0 < \pi$. The distance d is to the closest vertex or closest other side, whichever is closer. Here $\partial\Omega$ is in bold, and the sector S of radius d which is contained in Ω is in dashes.

Proof. For a boundary point $p_0 \in \partial\Omega$, p_0 either lies in the interior of a flat face, or at a corner. If at a corner, it can either be a convex or concave corner. But it turns out that the proof works more or less the same for all three cases. In fact, the proof is nearly identical to the proof of the interior case. Let θ_0 be the angle of $\partial\Omega$ at p_0, measured from the interior of Ω. If $0 < \theta_0 < \pi$, p_0 lies at a convex corner, and if $\pi < \theta_0 < 2\pi$, p_0 lies at a concave corner. If $\theta_0 = \pi$, then p_0 lies on the interior of a face of $\partial\Omega$.

Translate so that $p_0 = 0$ and rotate so that locally near 0

$$\Omega = \{(r, \theta) : 0 \leqslant r \leqslant R, \ -\theta_0/2 \leqslant \theta \leqslant \theta_0/2\}.$$

This just means that locally Ω looks like a sector of a disc, with Ω on the right hand side.

The proof procedes by considering the different boundary conditions when $\theta_0 = \pi$ and when $\theta_0 \neq \pi$. The first computation uses Lemma 2.1:

$$\int_\Omega \varphi([-h^2\Delta - 1, X]u)\bar{u}dV = -2\int_\Omega \varphi(-h^2\Delta)\bar{u}dV = 2\int_\Omega \varphi|u|^2 dV.$$

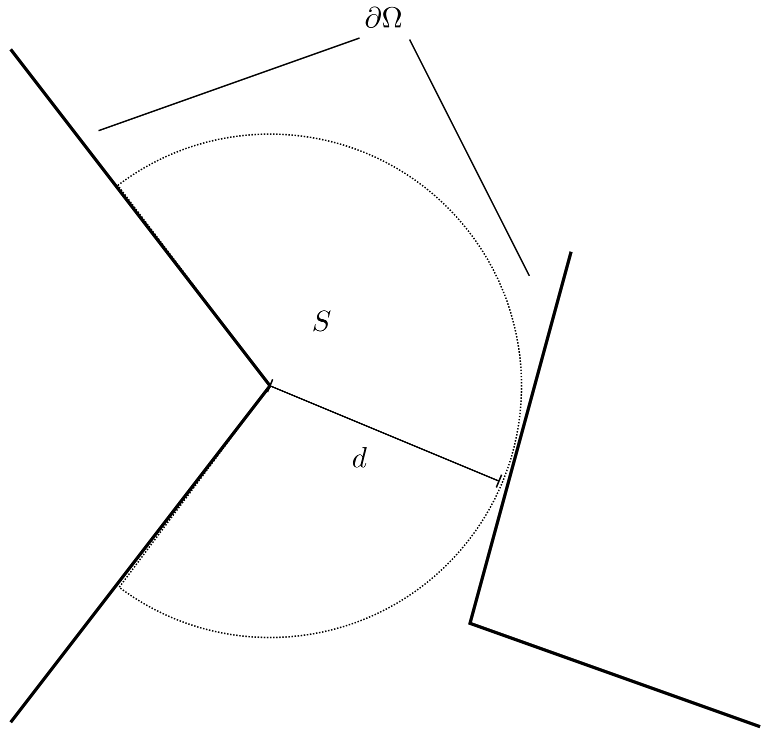

FIGURE 4. The polygon Ω near $p_0 \in \partial\Omega$ when p_0 lies at a concave corner. The domain Ω is to the right of boundary segments adjacent to p_0, so that the interior angle θ_0 satisfies $\pi < \theta_0 < 2\pi$. The distance d is to the closest vertex or closest other side, whichever is closer. Here $\partial\Omega$ is in bold, and the sector S of radius d which is contained in Ω is in dashes.

On the other hand, unpacking the commutator gives

$$\int_\Omega \varphi((-h^2\Delta - 1)Xu - X(-h^2\Delta - 1)u)\bar{u}dV = \int_\Omega \varphi((-h^2\Delta - 1)Xu)\bar{u}dV$$

since u satisfies (2.1). Using Green's formula gives

$$\begin{aligned}
&\int_\Omega \varphi((-h^2\Delta - 1)Xu)\bar{u}dV \\
&\quad = \int_\Omega (Xu)((-h^2\Delta - 1)\varphi\bar{u})dV - \int_{\partial\Omega} \varphi(h\partial_\nu hXu)\bar{u}dS + \int_{\partial\Omega} (hXu)(h\partial_\nu \varphi\bar{u})dS \\
&\quad = \int_\Omega (Xu)([-h^2\Delta, \varphi]\bar{u})dV - \int_{\partial\Omega} \varphi(h\partial_\nu hXu)\bar{u}dS + \int_{\partial\Omega} (hXu)(h\partial_\nu \varphi\bar{u})dS.
\end{aligned}$$

Here the boundary integrals over $\partial\Omega$ are really just over the segments within the support of φ.

We first consider the case $\theta_0 = \pi$. In this case, u lives in a flat face, so has the same boundary condition on the whole segment $F := \{x = 0,\ -R \leqslant y \leqslant R\}$ in rectangular coordinates. The outward unit normal derivative on F is $-\partial_x$. If $u|_F = 0$, then

$$\int_{\partial\Omega} \varphi(h\partial_\nu hXu)\bar{u}dS = 0.$$

If $u|_F = 0$, then $\partial_y u|_F = 0$ as well, since it is a tangential derivative. Writing $X = x\partial_x + y\partial_y$, we have

$$hXu|_F = (x\partial_x u + y\partial_y u)|_F = x\partial_x u|_F = 0,$$

since $F \subset \{x = 0\}$. Hence

$$\int_{\partial\Omega} (hXu)(h\partial_\nu \varphi \bar{u})dS = 0,$$

which proves the Lemma in this case.

If $\partial_\nu u|_F = 0$, then

$$\int_{\partial\Omega} (hXu)(h\partial_\nu \varphi \bar{u})dS = 0.$$

This is true since φ is assumed radial, so that $\partial_\theta \varphi = 0$, and along F we have $\partial_\nu = \partial_\theta = -\partial_x$. Since $\partial_\nu = -\partial_x$, that also means that $\partial_x u|_F = 0$. We now compute:

$$\begin{aligned} h\partial_\nu hXu|_F &= -h\partial_x(xh\partial_x u + yh\partial_y u)|_F \\ &= -hh\partial_x u|_F - xh^2\partial_x^2 u|_F - yh\partial_y h\partial_x u|_F \\ &= 0, \end{aligned}$$

since $\partial_x u = 0$, $x = 0$, and ∂_y is purely tangential.

We now consider the case when $0 < \theta_0 < \pi$, or $\pi < \theta_0 < 2\pi$ so that p_0 lies at a convex corner, or concave corner respectively. The upper (respectively lower) segments meeting at p_0 can be parametrized by $y = ax/b$ (respectively $y = -ax/b$). In the case $0 < \theta_0 < \pi$, $a, b > 0$, while in the case $\pi < \theta_0 < 2\pi$, $a > 0$ and $b < 0$. The reason for taking $b < 0$ instead of $a < 0$ here is so that the same notation may be used for the normal derivatives in both the convex and concave cases. Let F_1 denote the upper segment and F_2 denote the lower segment. There are three cases to consider, corresponding to the three possible boundary conditions (Dirichlet-Dirichlet, Neumann-Neumann, and Dirichlet-Neumann). Of course the Neumann-Dirichlet case is then obtained by reflection over $\{y = 0\}$. Our first task is to determine the correct outward pointing normal derivatives, and a choice of tangent derivatives. Along F_1, a choice of tangent derivative is

$$\partial_\tau = \frac{b}{c}\partial_x + \frac{a}{c}\partial_y,$$

and the outward normal derivative is

$$\partial_\nu = -\frac{a}{c}\partial_x + \frac{b}{c}\partial_y.$$

Here we pause briefly to note that this is where it is convenient to take $a > 0$, $b < 0$ in the concave case so that the normal derivative points outward. Along F_2, we have

$$\partial_\tau = \frac{b}{c}\partial_x - \frac{a}{c}\partial_y,$$

and

$$\partial_\nu = -\frac{a}{c}\partial_x - \frac{b}{c}\partial_y.$$

In the case where $u|_{F_j} = 0$ for $j = 1, 2$, we could use that $r\partial_r$ is tangential, but it is instructive to recall the argument from [Chr17] in rectangular coordinates. If $u|_{F_1} = 0$, then

$$0 = \partial_\tau u|_{F_1} = \frac{b}{c}\partial_x u|_{F_1} + \frac{a}{c}\partial_y u|_{F_1},$$

so that

$$\partial_x u|_{F_1} = -\frac{a}{b}\partial_y u|_{F_1}.$$

That means that

$$\begin{aligned}\partial_\nu u|_{F_1} &= \left(\frac{a^2}{bc} + \frac{b}{c}\right)\partial_y u|_{F_1} \\ &= \frac{c}{b}\partial_y u|_{F_1},\end{aligned}$$

so that

$$\partial_y u|_{F_1} = \frac{b}{c}\partial_\nu u|_{F_1},$$

and

$$\partial_x u|_{F_1} = -\frac{a}{c}\partial_\nu u|_{F_1}.$$

Plugging into the vector field X, we have

$$\begin{aligned}hXu|_{F_1} &= xh\partial_x u|_{F_1} + yh\partial_y u|_{F_1} \\ &= -\frac{a}{c}xh\partial_\nu u|_{F_1} + \left(\frac{ax}{b}\right)\frac{b}{c}h\partial_\nu u|_{F_1} \\ &= 0.\end{aligned}$$

If $u|_{F_2} = 0$, a similar computation holds on F_2, so that

$$-\int_{\partial\Omega}\varphi(h\partial_\nu hXu)\bar{u}dS + \int_{\partial\Omega}(hXu)(h\partial_\nu\varphi\bar{u})dS = 0,$$

and the Lemma is proved for that case.

For the next case, let us assume $\partial_\nu u|_{F_1} = 0$. Then

$$\begin{aligned}\int_{\partial\Omega}(hXu)(h\partial_\nu\varphi\bar{u})dS &= \int_{\partial\Omega}(hXu)(h\partial_\nu\varphi)\bar{u}dS \\ &= 0\end{aligned}$$

again since φ is assumed radial. On the other hand, for the other boundary integral term, we again compute:

$$\begin{aligned}h\partial_\nu hXu|_{F_1} &= \left(-\frac{a}{c}h\partial_x + \frac{b}{c}h\partial_y\right)(xh\partial_x u|_{F_1} + yh\partial_y u|_{F_1}) \\ &= -h\frac{a}{c}h\partial_x u|_{F_1} + h\frac{b}{c}h\partial_y u|_{F_1} \\ &\quad + xh\partial_\nu h\partial_x u|_{F_1} + \left(\frac{ax}{b}\right)h\partial_\nu h\partial_y u|_{F_1}.\end{aligned} \tag{4.2}$$

Here we have simply computed the contribution when $h\partial_\nu$ falls on x or on y in hX. Next, we write $h\partial_x u$ and $h\partial_y u$ in terms of tangential and normal derivatives. We have

$$\partial_x = \frac{b}{c}\partial_\tau - \frac{a}{c}\partial_\nu,$$

and

$$\partial_y = \frac{a}{c}\partial_\tau + \frac{b}{c}\partial_\nu.$$

Then

$$\begin{aligned} h\partial_\nu h\partial_x u|_{F_1} &= \frac{b}{c}h\partial_\nu h\partial_\tau u|_{F_1} - \frac{a}{c}h^2\partial_\nu^2 u|_{F_1} \\ &= \frac{a}{c}(1+h^2\partial_\tau^2)u|_{F_1}, \end{aligned}$$

since our change of variables $(x, y) \to (\tau, \nu)$ then gives

$$-h^2\partial_\nu^2 u - h^2\partial_\tau^2 u = u$$

in a neighbourhood of F_1 and $\partial_\tau\partial_\nu u|_{F_1} = 0$. Similarly,

$$\begin{aligned} h\partial_\nu h\partial_y u|_{F_1} &= \frac{a}{c}h\partial_\nu h\partial_\tau u|_{F_1} + \frac{b}{c}h^2\partial_\nu^2 u|_{F_1} \\ &= -\frac{b}{c}(1+h^2\partial_\tau^2)u|_{F_1}. \end{aligned}$$

Plugging in to (4.2), we have

$$\begin{aligned} h\partial_\nu hXu|_{F_1} &= -h\frac{a}{c}h\partial_x u|_{F_1} + h\frac{b}{c}h\partial_y u|_{F_1} \\ &\quad x\frac{a}{c}(1+h^2\partial_\tau^2)u|_{F_1} - \frac{b}{c}\left(\frac{ax}{b}\right)(1+h^2\partial_\tau^2)u|_{F_1} \\ &= -h\frac{a}{c}h\partial_x u|_{F_1} + h\frac{b}{c}h\partial_y u|_{F_1}. \end{aligned}$$

For the two remaining "lower order" terms, it is tempting to integrate and try to apply known restriction estimates. However, the lack of smoothness of the boundary makes this very delicate. In any case, we can still write these terms in terms of the normal and tangent derivatives, recalling that we are assuming that $\partial_\nu u = 0$. That means that

$$\begin{aligned} h\partial_x u|_{F_1} &= \left(\frac{b}{c}h\partial_\tau - \frac{a}{c}h\partial_\nu\right)u|_{F_1} \\ &= \frac{b}{c}h\partial_\tau u|_{F_1}. \end{aligned}$$

Similarly,

$$h\partial_y u|_{F_1} = \frac{a}{c}h\partial_\tau u|_{F_1}.$$

Then

$$\begin{aligned} -h\frac{a}{c}h\partial_x u|_{F_1} + h\frac{b}{c}h\partial_y u|_{F_1} &= h\left(-\frac{a}{c}\left(\frac{b}{c}h\partial_\tau u|_{F_1}\right) + \frac{b}{c}\left(\frac{a}{c}h\partial_\tau u|_{F_1}\right)\right) \\ &= 0. \end{aligned}$$

Again, a similar computation holds on F_2, so the Lemma is proved for the case where $\partial_\nu u = 0$ on both F_1 and F_2. Of course these analyses are completely independent of each other, so work for mixed boundary conditions as well. This proves the Lemma for p_0 at a corner, completing the proof.

□

We now continue with the proof of the Proposition.

Proof of Proposition 4.1. The proof is by contradiction, and proceeds almost verbatim from the interior case. We only point out the differences here.

Let $d = \mathrm{dist}\,(p_0, F')$, where F' is the nearest edge to p_0 which is not adjacent to p_0 as in the Proposition. This is either the nearest corner to p_0, or a different side if it is closer than the nearest corner. By rotating and translating Ω we may assume that $p_0 = 0$. We may also assume that Ω is locally symmetric over the line $y = 0$, and locally the segment $\{x > 0, \, y = 0\} \subset \Omega$. In the case p_0 is in the interior of a face, then the face is vertical and Ω locally lies to the right of that face. If p_0 lies at a convex corner making interior angle θ_0, then Ω lies locally to the right of p_0 and locally the angles above and below $\{y = 0\}$ are $\theta_0/2$. If p_0 lies at a concave corner, we use the same setup with angle $\theta_0/2$ above and below the line $\{y = 0\}$, but now Ω lies to the right of the segments at angle $\pm\theta_0/2$. See Figures 2-4.

Let $d, \epsilon, \varphi, \psi$ all be exactly the same as in the proof of Proposition 3.1. Every computation is exactly the same as in the proof for Proposition 3.1, except for the integrations by parts, since now there are potentially boundary terms. We now apply Lemma 4.2 to compute

$$\begin{aligned} 2\int_\Omega \varphi |u|^2 dV &= \int_\Omega \varphi([-h^2\Delta - 1, r\partial_r]u)\bar{u}dV \\ &= \int_\Omega (Xu)([-h^2\Delta, \varphi]\bar{u})dV. \end{aligned}$$

The rest of the proof follows precisely the proof of Proposition 3.1.

□

Appendix A. Construction of the function φ_1

In the proofs of Propositions 3.1 and 4.1, we have used a cutoff function with nice properties. The construction of such a function is more or less well-known, but we want very precise estimates, so we discuss the existence of such a cutoff in the next Lemma.

Lemma A.1. *Fix* $0 < \delta_1 < \delta_2$ *and* $0 < \epsilon \leqslant \min((\delta_2 - \delta_1)/4, 1/2)$. *There exists a function* $\varphi_1 \in \mathcal{C}^\infty(\mathbb{R})$ *satisfying the following conditions:*

(1) $\varphi_1 \geqslant 0$ *and* $\varphi' \leqslant 0$,
(2) $\varphi_1(s) \equiv 1$ *for* $s \leqslant \delta_1 + \epsilon^3$,
(3) $\varphi_1(s) \equiv 0$ *for* $s \geqslant \delta_2 - \epsilon^3$, *and*
(4) $|\varphi_1'| \leqslant \frac{1}{\delta_2 - \delta_1} + \epsilon$.

Proof. The only non-trivial part is the last condition on the derivative, since φ_1' is supported in a set strictly smaller than size $\delta_2 - \delta_1$. The idea is that we can make φ_1

as close to linear as we want and the only thing to work out is the dependence on the parameter $\epsilon > 0$. We recall that for any small number $\eta > 0$, there exists a smooth function $\varphi_2 : \mathbb{R} \to \mathbb{R}$ with $\varphi_2(s) \equiv 1$ for $s \leqslant 0$, $\varphi_2(s) \equiv 0$ for $s \geqslant 1$, $\varphi_2' \leqslant 0$, and $|\varphi_2'| \leqslant 1 + \eta$. For our $\epsilon > 0$, choose such a φ_2 with $\eta = \epsilon^2$.

In order to simplify notation, let $\delta = \delta_2 - \delta_1 > 0$. Let

$$\varphi_1(s) = \varphi_2\left(\frac{s - (\delta_1 + \epsilon^3)}{\delta - 2\epsilon^3}\right),$$

so that φ_1 satisfies $\varphi_1(s) \equiv 1$ for $s \leqslant \delta_1 + \epsilon^3$, $\varphi_1(s) \equiv 0$ for $s \geqslant \delta_2 - \epsilon^3$, $\varphi_1' \leqslant 0$, and

$$\begin{aligned}\sup |\varphi_1'| &\leqslant \frac{1}{\delta - 2\epsilon^3} \sup |\varphi_2'| \\ &\leqslant \frac{1+\epsilon^2}{\delta - 2\epsilon^3}.\end{aligned}$$

We are going to make a geometric series type expansion for which we need an upper bound. Recall that for $f(t) = (1-t)^{-1}$, $t \geqslant 0$ small, we have $f(t) = 1 + tf'(s)$ for some $0 \leqslant s \leqslant t$. As $f'(s) = (1-s)^{-2}$, we know that for $0 \leqslant s \leqslant t$, $|f'(s)| \leqslant (1-t)^{-2}$. Hence

$$f(t) \leqslant 1 + t(1-t)^{-2}. \tag{A.1}$$

Now we have assumed that $\epsilon \leqslant \min(\delta/4, 1/2)$, so we have

$$\frac{2\epsilon^3}{\delta} \leqslant \frac{\epsilon^2}{2} \leqslant \frac{1}{2},$$

so that

$$\frac{1}{1 - 2\epsilon^3/\delta} \leqslant 2.$$

Plugging this into (A.1) with $t = 2\epsilon^3/\delta$, we have

$$\begin{aligned}\frac{1+\epsilon^2}{\delta - 2\epsilon^3} &= \frac{1+\epsilon^2}{\delta}\left(\frac{1}{1 - 2\epsilon^3/\delta}\right) \\ &\leqslant \frac{1+\epsilon^2}{\delta}\left(1 + \frac{2\epsilon^3}{\delta}\left(\frac{1}{1-2\epsilon^3/\delta}\right)^2\right) \\ &\leqslant \frac{1+\epsilon^2}{\delta}\left(1 + \frac{8\epsilon^3}{\delta}\right).\end{aligned}$$

Continuing, and using that $\epsilon \leqslant \min(\delta/4, 1/2)$, we have

$$\begin{aligned}
\frac{1+\epsilon^2}{\delta}\left(1+\frac{8\epsilon^3}{\delta}\right) &\leqslant (1+\epsilon^2)\left(\frac{1}{\delta}+\frac{8\epsilon^3}{\delta^2}\right) \\
&\leqslant (1+\epsilon^2)\left(\frac{1}{\delta}+\frac{\epsilon}{2}\right) \\
&= \frac{1}{\delta}+\frac{\epsilon}{2}+\frac{\epsilon^2}{\delta}+\frac{\epsilon^3}{2} \\
&\leqslant \frac{1}{\delta}+\epsilon \\
&= \frac{1}{\delta_2-\delta_1}+\epsilon. \qquad (A.2)
\end{aligned}$$

□

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