EQUIDISTRIBUTION OF NEUMANN DATA MASS ON TRIANGLES

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ABSTRACT. In this paper we study the behaviour of the Neumann data of Dirichlet eigenfunctions on triangles. We prove that the L^2 norm of the (semi-classical) Neumann data on each side is equal to the length of the side divided by the area of the triangle. The novel feature of this result is that it is *not* an asymptotic, but an exact formula. The proof is by simple integrations by parts.

1. INTRODUCTION

Given a compact surface or manifold with boundary, it is an interesting question to consider restrictions of eigenfunctions to hypersurfaces; either the Dirichlet data or Neumann data (or both, the Cauchy data) can be considered. Perhaps the simplest question is to consider boundary values. That is, if we consider Dirichlet (respectively Neumann) eigenfunctions, we may try to study the Neumann (respectively Dirichlet) data on the boundary.

In this short note, we consider one of the simplest planar domains, a planar triangle T. Our main result is that the L^2 mass of the semi-classical Neumann data on each side of T *equals* the length of the side divided by the area of T. It should be emphasized that these formulae are equalities, not asymptotics or estimates.

Theorem 1. *Let T be a planar triangle with sides A, B, C, of lengths a, b, c respectively. Consider the (semi-classical) Dirichlet eigenfunction problem:*

$$\begin{cases} (-h^2\Delta - 1)u = 0, & \text{in } T, \\ u|_{\partial T} = 0, \end{cases} \tag{1.1}$$

and assume the eigenfunctions are normalized $\|u\|_{L^2(T)} = 1$.

Then the (semi-classical) Neumann data on the boundary satisfies

$$\int_A |h\partial_\nu u|^2 dS = \frac{a}{Area(T)},$$

$$\int_B |h\partial_\nu u|^2 dS = \frac{b}{Area(T)},$$

and

$$\int_C |h\partial_\nu u|^2 dS = \frac{c}{Area(T)},$$

where $h\partial_\nu$ is the semi-classical normal derivative on ∂T, dS is the arclength measure, and $Area(T)$ is the area of the triangle T.

We pause to note briefly that the the semiclassical parameter h takes discrete values as $h \to 0$ (reciprocals of eigenvalues).

Remark 1.1. We are calling this "equidistribution" of Neumann mass since it says that the Neumann data has the same mass to length ratio. Of course it does not say anything about local equidistribution to subsets of the sides.

To the author's knowledge, no exact formula such as this exists in the previous literature, except in cases where explicit formulae for the eigenfunctions are known. Even in these cases, the formulae typically depend on h. A statement such as Theorem 1 is false in general for other planar polygons. See Section 3 for the example of a square.

In order to better understand these formulae, in subsequent works, the author will study non-Euclidean triangles, and higher dimensional problems, as well as explore weaker lower bounds and interior lower bounds in some simple polygons.

1.1. **History.** Previous results on restrictions primarily focused on upper bounds. In the paper of Burq-Gérard-Tzvetkov [BGT07], restrictions of the Dirichlet data to arbitrary hypersurfaces were considered. An upper bound of the norm (squared) of the restrictions of $\mathcal{O}(h^{-1/2})$ was proved, and shown to be sharp. Of course this shows that there are *some* eigenfunctions with a known lower bound on the norms of restrictions. In the author's paper with Hassell-Toth [CHT13], an upper bound of $\mathcal{O}(1)$ was proved for (semi-classical) Neumann data restricted to arbitrary hypersurfaces, and also shown to be sharp. Again, this gives a lower and upper bound for *some* eigenfunctions.

In the case of quantum ergodic eigenfunctions, a little more is known. In the papers of Gérard-Leichtnam [GL93] and Hassell-Zelditch [HZ04], the Neumann (respectively Dirichlet) boundary data of Dirichlet (respectively Neumann) quantum ergodic eigenfunctions is studied, and shown to have an asymptotic formula for a *density one* subsequence. That means that there is a lower bound, and explicit local asymptotic formula in this special case, at least for most of the eigenfunctions. Similar statements were proved for interior hypersurfaces by Toth-Zelditch [TZ12, TZ13]. Again, potentially a sparse subsequence may behave differently. In the the author's paper with Toth-Zelditch [CTZ13], an asymptotic formula for the whole weighted Cauchy data is proved for the entire sequence of quantum ergodic eigenfunctions, however it is impossible to separate the behaviour of the Dirichlet versus Neumann data.

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2. Proof of Theorem 1

Assume the sides A, B, C are listed in clockwise orientation. We assume that A is the shortest side, followed by B and C with respective lengths $a \leqslant b \leqslant c$.

We use rectangular coordinates (x, y) in the plane, and orient our triangle so that the corner between B and C is at the origin $(0, 0)$. We further assume that the side A is parallel to the y axis.

We break our analysis into the two cases of acute triangles (including right triangles) and obtuse. See Figures 1 and 2 for a picture of the setup.

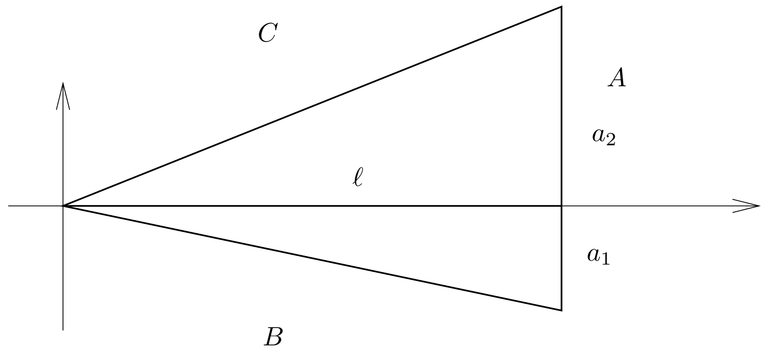

FIGURE 1. Setup for acute (and right) triangles

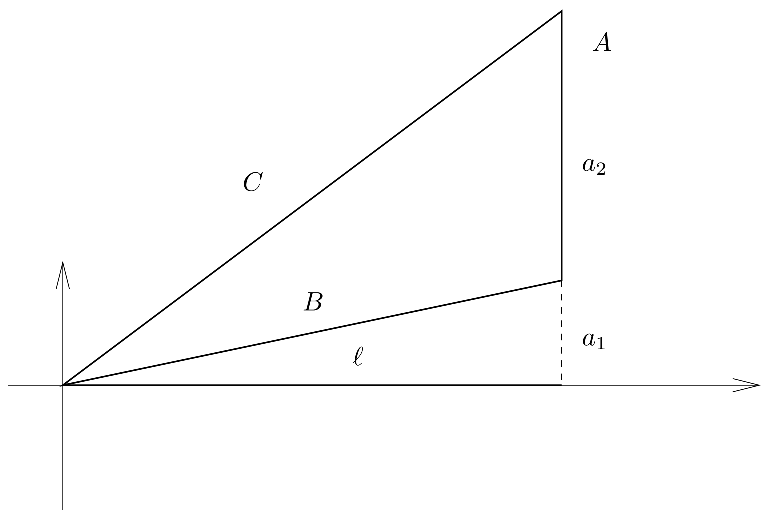

FIGURE 2. Setup for obtuse triangles

2.1. **Acute triangles.** Let ℓ be the segment on the x axis beginning at $(0,0)$ and perpendicular to the side A. Of course ℓ can be computed in terms of the sides, but its value is not necessary for this computation, other than to note that the area of T is $a\ell/2$. Write $A = A_1 \cup A_2$, where A_1 is the part of A under the x axis and A_2 is the part above. Let a_1, a_2 denote the respective sidelengths.

We can parametrize B and C with respect to x.

$$C = \left\{ (x,y) \in \mathbb{R}^2 : y = \frac{a_2}{\ell} x,\ 0 \leqslant x \leqslant \ell \right\},$$

and

$$B = \left\{ (x,y) \in \mathbb{R}^2 : y = -\frac{a_1}{\ell} x,\ 0 \leqslant x \leqslant \ell \right\}.$$

Then the arclength parameters are

$$\gamma_C = (1 + (a_2/\ell)^2)^{1/2} = \frac{(\ell^2 + a_2^2)^{1/2}}{\ell} = \frac{c}{\ell},$$

and

$$\gamma_B = (1 + (a_1/\ell)^2)^{1/2} = \frac{(\ell^2 + a_1^2)^{1/2}}{\ell} = \frac{b}{\ell},$$

and the unit tangent vectors are

$$\tau_C = \left(1, \frac{a_2}{\ell}\right) \gamma_C^{-1} = \left(\frac{\ell}{c}, \frac{a_2}{c}\right)$$

and

$$\tau_B = \left(1, -\frac{a_1}{\ell}\right) \gamma_B^{-1} = \left(\frac{\ell}{b}, -\frac{a_1}{b}\right).$$

From this we have the outward unit normal vectors

$$\nu_C = \left(-\frac{a_2}{c}, \frac{\ell}{c}\right)$$

and

$$\nu_B = \left(-\frac{a_1}{b}, -\frac{\ell}{b}\right).$$

Of course the outward normal to A is $\nu_A = (1, 0)$.

We are assuming Dirichlet boundary conditions, which implies that the tangential derivatives of u vanish on ∂T. That is,

$$\partial_y u = 0$$

on A, and

$$\tau_C \cdot \nabla u = \frac{\ell}{c} \partial_x u + \frac{a_2}{c} \partial_y u = 0$$

on C. Similarly,

$$\tau_B \cdot \nabla u = \frac{\ell}{b} \partial_x u - \frac{a_1}{b} \partial_y u = 0$$

along B. Rearranging, we have

$$h \partial_x u = -\frac{a_2}{\ell} h \partial_y u$$

on C and

$$h \partial_x u = \frac{a_1}{\ell} h \partial_y u$$

on B.

Making the substitutions, along C we have

$$\begin{aligned} h\partial_{\nu_C} u &= \nu_C \cdot h\nabla u \\ &= -\frac{a_2}{c} h\partial_x u + \frac{\ell}{c} h\partial_y u \\ &= \left(\frac{a_2^2}{c\ell} h\partial_x u + \frac{\ell}{c} \right) h\partial_y u \\ &= \left(\frac{a_2^2 + \ell^2}{c\ell} \right) h\partial_y u \\ &= \frac{c}{\ell} h\partial_y u. \end{aligned}$$

Hence

$$h\partial_y u = \frac{\ell}{c} h\partial_{\nu_C} u$$

on C. Substituting again, we have

$$h\partial_x u = -\frac{a_2}{\ell} h\partial_y u = -\frac{a_2}{c} h\partial_{\nu_C} u$$

along C. Similarly, along B, we have

$$h\partial_y u = -\frac{\ell}{b} h\partial_{\nu_B} u$$

and

$$h\partial_x u = -\frac{a_1}{b} h\partial_{\nu_B} u.$$

We now consider the vector field

$$X = (x+m)\partial_x + (y+n)\partial_y,$$

where m, n are parameters independent of x and y. Since $m\partial_x$ commutes with $-h^2\Delta$ as well as $n\partial_y$, the usual computation yields

$$[-h^2\Delta - 1, X] = -2h^2\Delta.$$

Then using eigenfunction equation (1.1), we have

$$\begin{aligned} \int_T ([-h^2\Delta - 1, X]u)\bar{u} dV &= -2 \int_T (h^2\Delta u)\bar{u} dV \\ &= \int_T 2|u|^2 dV \\ &= 2, \end{aligned}$$

since u is normalized.

On the other hand, again using the eigenfunction equation (1.1) again, we have

$$\begin{aligned} &\int_T ([-h^2\Delta - 1, X]u)\bar{u} dV \\ &\quad = \int_T ((-h^2\Delta - 1)Xu)\bar{u} dV - \int_T (X(-h^2\Delta - 1)u)\bar{u} dV \\ &\quad = \int_T ((-h^2\Delta - 1)Xu)\bar{u} dV. \end{aligned}$$

Integrating by parts and using the eigenfunction equation and Dirichlet boundary conditions, we have

$$\begin{aligned}
&\int_T ((-h^2\Delta - 1)Xu)\bar{u}dV \\
&\quad = \int_T (Xu)((-h^2\Delta - 1))\bar{u}dV \\
&\qquad - \int_{\partial T} (h\partial_\nu hXu)\bar{u}dS + \int_{\partial T} (hXu)(h\partial_\nu \bar{u})dS \\
&\quad = \int_{\partial T} (hXu)(h\partial_\nu \bar{u})dS,
\end{aligned}$$

Hence we have computed:

$$2 = \int_{\partial T} (hXu)(h\partial_\nu \bar{u})dS.$$

Let us break up the analysis into the three different sides. In order to simplify notation somewhat, set

$$I_A = \int_A |h\partial_\nu u|^2 dS,$$

and similarly for B and C. Notice we have left the surface measure dS alone, even though we could write it explicitly in terms of the arclength parameters computed above. This is not necessary for the analysis.

Returning now to our computations of the normal derivatives, we have

$$\begin{aligned}
&\int_A (hXu)(h\partial_\nu \bar{u})dS \\
&\quad = \int_A (((x+m)h\partial_x + (y+n)h\partial_y)u)(h\partial_\nu \bar{u})dS \\
&\quad = (\ell + m)I_A,
\end{aligned}$$

since $x = \ell$ on A.

Continuing, using that along C, we have $y = (a_2/\ell)x$:

$$\begin{aligned}
&\int_C (hXu)(h\partial_\nu \bar{u})dS \\
&\quad = \int_C (((x+m)h\partial_x + (y+n)h\partial_y)u)(h\partial_\nu \bar{u})dS \\
&\quad = \int_C \left(((x+m)h\partial_x + \left(\frac{a_2}{\ell}x + n\right) h\partial_y)u\right)(h\partial_\nu \bar{u})dS \\
&\quad = \int_C \left(\left((x+m)\left(-\frac{a_2}{c}\right) + \left(\frac{a_2}{\ell}x + n\right)\left(\frac{\ell}{c}\right)\right) h\partial_{\nu_C} u\right)(h\partial_{\nu_C} \bar{u})dS \\
&\quad = \int_C \left(\left(-\frac{a_2}{c}m + \frac{\ell}{c}n\right) h\partial_{\nu_C} u\right)(h\partial_{\nu_C} \bar{u})dS \\
&\quad = \left(-\frac{a_2}{c}m + \frac{\ell}{c}n\right) I_C.
\end{aligned}$$

Similarly, along B, we have $y = -(a_1/\ell)x$:

$$\begin{aligned}
&\int_B (hXu)(h\partial_\nu \bar{u})dS \\
&\quad = \int_B (((x+m)h\partial_x + (y+n)h\partial_y)u)(h\partial_\nu \bar{u})dS \\
&\quad = \int_B \left(((x+m)h\partial_x + \left(-\frac{a_1}{\ell}x + n\right) h\partial_y)u\right) (h\partial_\nu \bar{u})dS \\
&\quad = \int_B \left(\left((x+m)\left(-\frac{a_1}{b}\right) + \left(-\frac{a_1}{\ell}x + n\right)\left(-\frac{\ell}{b}\right)\right) h\partial_{\nu_B} u\right) (h\partial_{\nu_B} \bar{u})dS \\
&\quad = \int_B \left(\left(-\frac{a_1}{b}m - \frac{\ell}{b}n\right) h\partial_{\nu_B} u\right) (h\partial_{\nu_B} \bar{u})dS \\
&\quad = \left(-\frac{a_1}{b}m - \frac{\ell}{b}n\right) I_B.
\end{aligned}$$

Summing up, we have:

$$2 = (\ell + m)I_A + \left(-\frac{a_1}{b}m - \frac{\ell}{b}n\right) I_B + \left(-\frac{a_2}{c}m + \frac{\ell}{c}n\right) I_C. \tag{2.1}$$

First, set $m = n = 0$. Then we get $2 = \ell I_A$, or

$$\begin{aligned}
I_A &= \frac{2}{\ell} \\
&= \frac{a}{a\ell/2} \\
&= \frac{a}{\text{Area}(T)}.
\end{aligned}$$

Now we observe that the left hand side of (2.1) is independent of m and n so we can differentiate with respect to m and n to get two new equations. That is, differentiating both sides of (2.1) with respect to m yields

$$0 = I_A - \frac{a_1}{b}I_B - \frac{a_2}{c}I_C,$$

and plugging in the value of I_A, we have

$$\frac{a_1}{b}I_B + \frac{a_2}{c}I_C = \frac{2}{\ell}. \tag{2.2}$$

Now differentiating (2.1) with respect to n, we have

$$0 = -\frac{\ell}{b}I_B + \frac{\ell}{c}I_C,$$

so that

$$I_B = \frac{b}{c}I_C.$$

Plugging in to (2.2), we have

$$\begin{aligned}\frac{2}{\ell} &= \left(\left(\frac{a_1}{b}\right)\left(\frac{b}{c}\right) + \frac{a_2}{c} \right) I_C \\ &= \left(\frac{a_1}{c} + \frac{a_2}{c}\right) I_C \\ &= \frac{a}{c} I_C.\end{aligned}$$

Hence

$$I_C = \frac{2c}{a\ell} = \frac{c}{\mathrm{Area}(T)},$$

and back substituting,

$$I_B = \frac{b}{c} I_C = \frac{b}{\mathrm{Area}(T)}.$$

This proves the theorem for acute and right triangles.

2.2. **Obtuse triangles.** The proof is nearly the same, with several sign changes. Using the setup in Figure 2, we have

$$C = \left\{ (x, y) : 0 \leqslant x \leqslant \ell, \text{ and } y = \frac{(a + a_1)}{\ell x} \right\},$$

and

$$B = \left\{ (x, y) : 0 \leqslant x \leqslant \ell, \text{ and } y = \frac{a_1}{\ell} x \right\}.$$

Similar computations as above lead to the following: along C,

$$h\partial_y u = \frac{\ell}{c} h\partial_{\nu_C} u,$$

and

$$h\partial_x u = -\frac{a + a_1}{c} h\partial_{\nu_C}.$$

Along B we have

$$h\partial_y u = -\frac{\ell}{b} h\partial_{\nu_B} u$$

and

$$h\partial_x u = \frac{a_1}{b} h\partial_{\nu_B} u.$$

The same commutator computation holds, and similar substitutions as in the acute case yield the equation

$$2 = (\ell + m) I_A + \left(\frac{a_1}{b} m - \frac{\ell}{b} n \right) I_B + \left(-\frac{a + a_1}{c} m + \frac{\ell}{c} n \right) I_C. \tag{2.3}$$

Again first setting $m = n = 0$, we get

$$I_A = \frac{2}{\ell} = \frac{a}{\mathrm{Area}(T)}.$$

Differentiating with respect to m and n we get the two equations

$$0 = I_A + \frac{a_1}{b} I_B - \frac{a + a_1}{c} I_C, \tag{2.4}$$

and

$$0 = -\frac{\ell}{b} I_B + \frac{\ell}{c} I_C,$$

so that again

$$I_B = \frac{b}{c} I_C.$$

Now substituting into (2.4), we have

$$\frac{2}{\ell} = \left(-\left(\frac{a_1}{b} \right) \left(\frac{b}{c} \right) + \frac{a + a_1}{c} \right) I_C,$$

or

$$I_C = \frac{2c}{a\ell} = \frac{c}{\text{Area}(T)}.$$

Back substituting once again, we have also

$$I_B = \frac{b}{\text{Area}(T)}.$$

This completes the proof of the obtuse triangle case, and hence proves the theorem.

3. Other polygons

Theorem 1 is false in general for other planar polygons. One can see from the computations above that it is straightforward to come up with 3 independent equations relating the Neumann data on the sides. The proof suggests that only three equations are possible in general. Of course we do not have a proof of that. However, even for convex polygons the theorem is false in general.

Consider the square $\Omega = [0, 2\pi]^2$. The Dirichlet eigenfunctions are given by the Fourier basis. Let us examine some specific choices. That is, for integers j, k let

$$u_{jk}(x, y) = (\pi)^{-1} \sin(jx) \sin(ky).$$

The u_{jk} vanish on $\partial\Omega$, and are normalized. We have

$$-\Delta u_{jk} = (j^2 + k^2) u_{jk}$$

as usual. Rescaling to a semi-classical equation, we take $h = (j^2 + k^2)^{-1/2}$ to get

$$-h^2 \Delta u_{jk} = u_{jk}.$$

On $x = 0$ and $x = 2\pi$ repectively, we have

$$h\partial_\nu u_{jk}|_{x=0} = \pi^{-1}(-j) h \sin(ky)$$

and

$$h\partial_\nu u_{jk}|_{x=2\pi} = \pi^{-1} jh \sin(ky).$$

Hence the norm of the Neumann data along either $x = 0$ or $x = 2\pi$ is

$$\int_0^{2\pi} |h\partial_\nu u_{jk}|_{x=0,2\pi}|^2 dy = \pi^{-1} h^2 j^2.$$

If $k \gg j$, we can make $h^2 j^2$ as small as we like, so in fact we can say only

$$\int_0^{2\pi} |h\partial_\nu u_{jk}|_{x=0,2\pi}|^2 dy \geqslant ch^2$$

for some $c > 0$. A rough conjecture is that a lower bound of h^2 holds for any polygon in the plane. The author plans to revisit this question in subsequence papers.

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