

UNIQUE CONTINUATION FOR QUASIMODES ON SURFACES OF REVOLUTION: ROTATIONALLY INVARIANT NEIGHBOURHOODS

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ABSTRACT. We prove a strong conditional unique continuation estimate for irreducible quasimodes in rotationally invariant neighbourhoods on compact surfaces of revolution. The estimate states that Laplace quasimodes which cannot be decomposed as a sum of other quasimodes have L^2 mass bounded below by $C_\epsilon \lambda^{-1-\epsilon}$ for any $\epsilon > 0$ on any open rotationally invariant neighbourhood which meets the semiclassical wavefront set of the quasimode. For an analytic manifold, we conclude the same estimate with a lower bound of $C_\delta \lambda^{-1+\delta}$ for some fixed $\delta > 0$.

1. INTRODUCTION

We consider a compact periodic surface of revolution $X = \mathbb{S}_x^1 \times \mathbb{S}_\theta^1$, equipped with a metric of the form

$$ds^2 = dx^2 + A^2(x)d\theta^2,$$

where $A \in C^\infty$ is a smooth function, $A \geq \epsilon > 0$. Our analysis is microlocal, so applies also to any compact surface of revolution with no boundary, and to certain surfaces of revolution with boundary under mild assumptions, however we will concentrate on the toral case for ease of exposition.

From such a metric, we get the volume form

$$d\text{Vol} = A(x)dx d\theta,$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x)f.$$

We are concerned with quasimodes, which are the building blocks from which eigenfunctions are made, however we need to define the most basic kind of quasimodes, which we will call *irreducible quasimodes*, meaning the quasimodes which cannot be decomposed as a sum of two or more nontrivial quasimodes. In order to make our definitions, we recall first that the geodesic flow on T^*X is the Hamiltonian system associated to the principal symbol of the Laplace-Beltrami operator:

$$p(x, \xi, \theta, \eta) = \xi^2 + A^{-2}(x)\eta^2.$$

A fixed energy level $p = \text{const.}$ consists of all the geodesics of that constant “speed”. For the case of the geodesic Hamiltonian system on T^*X , there are two conserved

The author is extremely grateful to András Vasy and Jared Wunsch for their help with the moment map definition of irreducible quasimode, and for their comments and suggestions on earlier versions of this paper. This work is supported in part by NSF grant DMS-0900524.

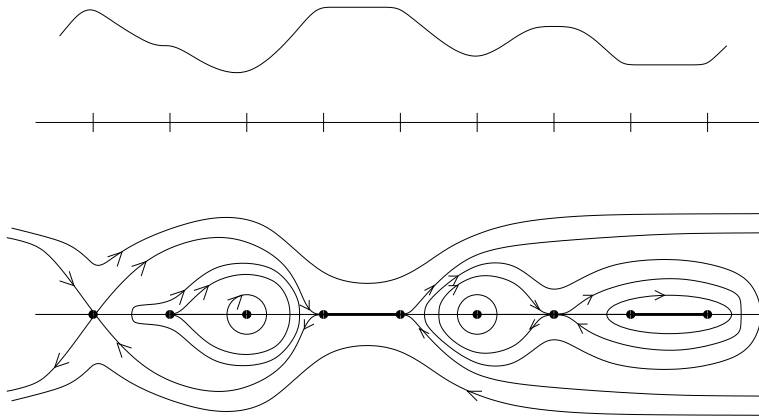


FIGURE 1. The reduced phase space of a toral surface of revolution with many periodic latitudinal geodesics.

quantities, the total energy and the angular momentum η^2 . The *moment map* is the map sending points of T^*X to their associated conserved quantities, that is

$$M(x, \xi, \theta, \eta) = \begin{pmatrix} \xi^2 + A^{-2}(x)\eta^2 \\ \eta^2 \end{pmatrix}.$$

When the gradient of M has rank 2, then M defines a submersion, so each connected component of the preimage is a 2-manifold. Points in T^*X where M has rank 1 or 0 are called critical points, and points $(P, Q) \in \mathbb{R}^2$ such that $\{M = (P, Q)\}$ contains critical points are called critical values. Critical points correspond to latitudinal periodic geodesics, which can also carry quasimode mass, and critical values have preimages which may have infinitely many latitudinal periodic geodesics. The semiclassical wavefront set is always a closed invariant subset of the energy surface, so our definition of irreducible quasimode will be one which has wavefront mass confined to the closure of one of these two kinds of sets, distinguished by rank of M .

Definition 1.1. An *irreducible* quasimode is a quasimode whose semiclassical wavefront set is contained in the closure of a single connected component in T^*X where the moment map has constant rank.

We also will require a limit on the geodesic complexity by assuming there are only a finite number of connected regions of latitudinal periodic geodesics. This will not preclude having infinitely many periodic latitudinal geodesics, but merely having accumulation points of connected components of latitudinal geodesics. We therefore will assume that the moment map has a finite number of critical values, each of which has a preimage of finitely many non-empty connected components. Note this allows intervals of latitudinal periodic geodesics, but does not allow accumulation of such sets. For an example, see Figure 1.

Finally, we will require a certain 0-Gevrey regularity on the manifold, which in a sense says our manifold is not too far from being analytic. Such a 0-Gevrey assumption nevertheless allows for non-trivial functions which are constant on intervals, so this is a very general class of manifolds. Of course this includes analytic

manifolds, for which we have a stronger estimate. See Subsection 2.1 for the precise definitions.

Theorem 1. *Let X be as above, for a generating curve in the 0-Gevrey class $A(x) \in \mathcal{G}_\tau^0(\mathbb{R})$ for some $\tau < \infty$. Assume the moment map has finitely many critical values, with preimages consisting of finitely many connected components. Suppose u is a (weak) irreducible quasimode satisfying $\|u\| = 1$ and*

$$(-\Delta - \lambda^2)u = \mathcal{O}(\lambda^{-\beta_0}),$$

for some fixed $\beta_0 > 0$. Let $\Omega \subset X$ be a rotationally invariant neighbourhood, $\Omega = (a, b)_x \times \mathbb{S}_\theta^1$. Then either

(1)

$$\|u\|_{L^2(\Omega)} = \mathcal{O}(\lambda^{-\infty}),$$

or

(2) for any $\epsilon > 0$, there exists $C = C_{\epsilon, \Omega, \beta_0} > 0$ such that

$$(1.1) \quad \|u\|_{L^2(\Omega)} \geq C\lambda^{-1-\epsilon}.$$

Remark 1.2. The proof will show that a more or less straightforward commutator/contradiction argument gives a lower bound of $\lambda^{-1-\beta_0}$. The difficulty comes in trying to beat this lower bound.

In the analytic category, we have a significant improvement. Of course in the case of an analytic manifold, there can be no infinitely degenerate critical elements, nor can there be any accumulation points of sets of latitudinal periodic geodesics, so we do not need to make the assumption about finite geodesic complexity.

Corollary 1.3. *Let X be as above, and assume X is analytic. Suppose u is a (weak) irreducible quasimode satisfying $\|u\| = 1$ and*

$$(-\Delta - \lambda^2)u = \mathcal{O}(1).$$

Then for any open rotationally invariant neighbourhood $\Omega \subset X$, either

(1)

$$\|u\|_{L^2(\Omega)} = \mathcal{O}(\lambda^{-\infty}),$$

or

(2) there exists a fixed $\delta > 0$ and a constant $C = C_\Omega > 0$ such that

$$\|u\|_{L^2(\Omega)} \geq C\lambda^{-1+\delta}.$$

Remark 1.4. The assumption that $\Omega \subset X$ is a rotationally invariant neighbourhood of the form $\Omega = (a, b)_x \times \mathbb{S}_\theta^1$ is necessary for this level of generality. To see this, consider the case where X has part of a 2-sphere embedded in it. Then there are many periodic geodesics close to the latitudinal one. But these geodesics can be rotated in θ without changing the angular momentum. Each one of these is elliptic and can carry a Gaussian beam type quasimode. Hence one can create an irreducible quasimode as a superposition of these Gaussian beams. The resulting “band” of quasimodes need not have nontrivial mass except in a rotationally invariant neighbourhood. See Figure 2.

2. PRELIMINARIES

In this section we review some of the definitions and preliminary computations necessary for Theorem 1, as well as recall the spectral estimates we will be using.

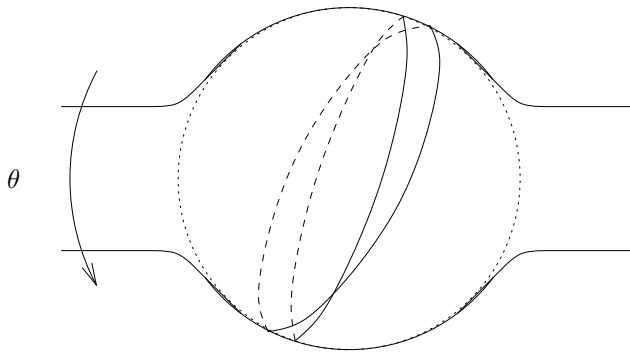


FIGURE 2. A surface of revolution with a piece of \mathbb{S}^2 embedded. Also sketched are two “isoenergetic” periodic geodesics which are θ rotations of each other. One can construct pathological quasimodes which are continuous, compactly supported superpositions of isoenergetic quasimodes associated to such geodesics.

2.1. The 0-Gevrey class of functions. For this paper, we use the following 0-Gevrey classes of functions with respect to order of vanishing, introduced in [Chr13].

Definition 2.1. For $0 \leq \tau < \infty$, let $\mathcal{G}_\tau^0(\mathbb{R})$ be the set of all smooth functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that, for each $x_0 \in \mathbb{R}$, there exists a neighbourhood $U \ni x_0$ and a constant C such that, for all $0 \leq s \leq k$,

$$|\partial_x^k f(x) - \partial_x^k f(x_0)| \leq C(k!)^C |x - x_0|^{-\tau(k-s)} |\partial_x^s f(x) - \partial_x^s f(x_0)|, \quad x \rightarrow x_0 \text{ in } U.$$

This definition says that the order of vanishing of derivatives of a function is only polynomially worse than that of lower derivatives. Every analytic function is in one of the 0-Gevrey classes \mathcal{G}_τ^0 for some $\tau < \infty$, but many more functions are as well. For example, the function

$$f(x) = \begin{cases} \exp(-1/x^p), & \text{for } x > 0, \\ 0, & \text{for } x \leq 0 \end{cases}$$

is in \mathcal{G}_{p+1}^0 , but

$$f(x) = \begin{cases} \exp(-\exp(1/x)), & \text{for } x > 0, \\ 0, & \text{for } x \leq 0 \end{cases}$$

is not in any 0-Gevrey class for finite τ .

2.2. Conjugation to a flat problem. We observe that we can conjugate Δ by an isometry of metric spaces and separate variables so that spectral analysis of Δ is equivalent to a one-variable semiclassical problem with potential. That is, let $T : L^2(X, d\text{Vol}) \rightarrow L^2(X, dx d\theta)$ be the isometry given by

$$Tu(x, \theta) = A^{1/2}(x)u(x, \theta).$$

Then $\tilde{\Delta} = T\Delta T^{-1}$ is essentially self-adjoint on $L^2(X, dx d\theta)$. A simple calculation gives

$$-\tilde{\Delta}f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x))f,$$

where the potential

$$V_1(x) = \frac{1}{2}A''A^{-1} - \frac{1}{4}(A')^2A^{-2}.$$

If we now separate variables and write $\psi(x, \theta) = \sum_k \varphi_k(x)e^{ik\theta}$, we see that

$$(-\tilde{\Delta} - \lambda^2)\psi = \sum_k e^{ik\theta} P_k \varphi_k(x),$$

where

$$P_k \varphi_k(x) = \left(-\frac{d^2}{dx^2} + k^2 A^{-2}(x) + V_1(x) - \lambda^2 \right) \varphi_k(x).$$

Setting $h = |k|^{-1}$ and rescaling, we have the semiclassical operator

$$(2.1) \quad P(z, h)\varphi(x) = \left(-h^2 \frac{d^2}{dx^2} + V(x) - z \right) \varphi(x),$$

where the potential is

$$V(x) = A^{-2}(x) + h^2 V_1(x)$$

and the spectral parameter is $z = h^2 \lambda^2$. In Section 3 we will at first let $h = \lambda^{-1}$ be our semiclassical parameter for the whole quasimode, but then switch to $h = |k|^{-1}$ to estimate the parts of the quasimode microsupported where the critical elements are located. The relevant microlocal estimates near critical elements are summarized in the following Subsection.

2.3. Spectral estimates for weakly unstable critical sets. In this subsection we summarize the spectral estimates we will use for weakly unstable critical elements obtained in [Chr07, Chr10, Chr11, CW11, CM13, Chr13].

Definition 2.2. Let (P, Q) be a critical value of the moment map. Then there are points in $M^{-1}(P, Q)$ where the moment map has rank 1 (or 0, but these points are easy to handle (see below)). For these points, there are latitudinal periodic geodesics. If the principal part of the potential, $A^{-2}(x)$, for the reduced Hamiltonian $\xi^2 + A^{-2}(x)$ has an ‘‘honest’’ minimum at x_0 in the sense that if $[a, b]$ is the maximal closed interval containing x_0 with $A^{-2}(x) = A^{-2}(x_0)$ on it, then $(A^{-2})' < 0$ for $x < a$ in some small neighbourhood, and $(A^{-2})' > 0$ for $x > b$ in some other small neighbourhood, then we say this critical element is *weakly stable*. In all other cases, we say the critical element is *weakly unstable*.

In the following subsections, we review the microlocal estimates from [Chr13] for weakly unstable critical elements. Taken together, they imply the following theorem.

Theorem 2. *Let Λ be a weakly unstable critical element in the reduced phase space $T^*\mathbb{S}_x^1$, and assume u has h -wavefront set sufficiently close Λ . Then for any $\epsilon > 0$, there exists $C = C_\epsilon$ such that*

$$\|u\| \leq Ch^{-2-\epsilon} \|((hD)^2 + V(x) - z)u\|,$$

for any $z \in \mathbb{R}$.

2.3.1. *Unstable nondegenerate critical elements.* A nondegenerate unstable critical element exists where the principal part of the potential $V_0(x) = A^{-2}(x)$ has a nondegenerate maximum. To say that $x = 0$ is a nondegenerate maximum means that $x = 0$ is a critical point of $V_0(x)$ satisfying $V_0'(0) = 0$, $V_0''(0) < 0$.

The following result as stated can be read off from [Chr07, Chr11], and has also been studied in slightly different contexts in [CdVP94a, CdVP94b] and [BZ04], amongst many others.

Lemma 2.3. *Suppose $x = 0$ is a nondegenerate local maximum of the principal part of the potential V_0 , $V_0(0) = 1$. For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(2.2) \quad \|P(z, h)\varphi^w u\| \geq C_\epsilon \frac{h}{\log(1/h)} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

Remark 2.4. This estimate is known to be sharp, in the sense that the logarithmic loss cannot be improved (see, for example, [CdVP94a]).

2.3.2. *Unstable finitely degenerate critical elements.* In this subsection, we consider an isolated critical point at an unstable but finitely degenerate maximum. That is, we now assume that $x = 0$ is a degenerate maximum for the function $V_0(x) = A^{-2}(x)$ of order $m \geq 2$. If we again assume $V_0(0) = 1$, then this means that near $x = 0$, $V_0(x) \sim 1 - x^{2m}$. Critical points of this form were first studied in [CW11].

This Lemma and the proof are given in [CW11, Lemma 2.3].

Lemma 2.5. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(2.3) \quad \|P(z, h)\varphi^w u\| \geq C_\epsilon h^{2m/(m+1)} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

Remark 2.6. This estimate is known to be sharp, in the sense that the exponent $2m/(m+1)$ cannot be improved (see [CW11]).

2.3.3. *Finitely degenerate inflection transmission critical elements.* We next study the case when the principal part of the potential has an inflection point of finitely degenerate type. That is, let us assume the point $x = 1$ is a finitely degenerate inflection point, so that locally near $x = 1$, the potential $V_0(x) = A^{-2}(x)$ takes the form

$$V_0(x) \sim C_1^{-1} - c_2(x-1)^{2m_2+1}, \quad m_2 \geq 1$$

where $C_1 > 1$ and $c_2 > 0$. Of course the constants are arbitrary (chosen to agree with those in [CM13]), and c_2 could be negative without changing much of the analysis. This Lemma and the proof are in [CM13].

Lemma 2.7. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x-1, \xi)| \leq \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$(2.4) \quad \|P(z, h)\varphi^w u\| \geq C_\epsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon].$$

Remark 2.8. This estimate is also known to be sharp in the sense that the exponent $(4m_2+2)/(2m_2+3)$ cannot be improved (see [CM13]).

2.3.4. *Unstable infinitely degenerate and cylindrical critical elements.* In this subsection, we study the case where the principal part of the potential $V(x) = A^{-2}(x) + h^2V_1(x)$ has an infinitely degenerate maximum, say, at the point $x = 0$. Let $V_0(x) = A^{-2}(x)$. As usual, we again assume that $V_0(0) = 1$, so that

$$V_0(x) = 1 - \mathcal{O}(x^\infty)$$

in a neighbourhood of $x = 0$. Of course this is not very precise, as V_0 could be constant in a neighbourhood of $x = 0$ and still satisfy this. So let us first assume that $V_0(0) = 1$, and $V_0'(x)$ vanishes to infinite order at $x = 0$, however, $\pm V_0'(x) < 0$ for $\pm x > 0$. That is, the critical point at $x = 0$ is infinitely degenerate but isolated.

Lemma 2.9. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x, \xi)| \leq \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon, \eta} > 0$ such that*

$$(2.5) \quad \|P(z, h)\varphi^w u\| \geq C_{\epsilon, \eta} h^{2+\eta} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

For our next result, we consider the case where there is a whole interval at a local maximum value. That is, we assume the principal part of the effective potential $V_0(x)$ has a maximum $V_0(x) \equiv 1$ on an interval, say $x \in [-a, a]$, and that $\pm V_0'(x) < 0$ for $\pm x > a$ in some neighbourhood.

Lemma 2.10. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x| \leq a + \epsilon, |\xi| \leq \epsilon\}$. Then for any $\eta > 0$, there exists $C_{\epsilon, \eta} > 0$ such that*

$$(2.6) \quad \|P(z, h)\varphi^w u\| \geq C_{\epsilon, \eta} h^{2+\eta} \|\varphi^w u\|, \quad z \in [1 - \epsilon, 1 + \epsilon].$$

2.3.5. *Infinitely degenerate and cylindrical inflection transmission critical elements.* In this subsection, we assume the effective potential has a critical element of infinitely degenerate or cylindrical inflection transmission type. This is very similar to Subsection 2.3.4, but now the potential is assumed to be monotonic in a neighbourhood of the critical value.

We begin with the case where the potential has an isolated infinitely degenerate critical point of inflection transmission type. As in the previous subsection, we write $V(x) = A^{-2}(x) + h^2V_1(x)$ and denote $V_0(x) = A^{-2}(x)$ to be the principal part of the potential. Let us assume the point $x = 1$ is an infinitely degenerate inflection point, so that locally near $x = 1$, the potential takes the form

$$V_0(x) \sim C_1^{-1} - (x - 1)^\infty,$$

where $C_1 > 1$. Of course the constant is arbitrary (chosen to again agree with those in [CM13]). Let us assume that our potential satisfies $V_0'(x) \leq 0$ near $x = 1$, with $V_0'(x) < 0$ for $x \neq 1$ in some neighbourhood so that the critical point $x = 1$ is isolated.

Lemma 2.11. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x - 1, \xi)| \leq \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon, \eta} > 0$ such that*

$$(2.7) \quad \|P(z, h)\varphi^w u\| \geq C_\epsilon h^{2+\eta} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon].$$

On the other hand, if $V_0'(x) \equiv 0$ on an interval, say $x - 1 \in [-a, a]$ with $V_0'(x) < 0$ for $x - 1 < -a$ and $x - 1 > a$, we do not expect anything better than Lemma 2.11. The next lemma says that this is exactly what we do get. To fix an energy level, assume $V_0 \equiv C_1^{-1}$ on $[-a, a]$.

Lemma 2.12. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|x-1| \leq a + \epsilon, |\xi| \leq \epsilon\}$. Then for any $\eta > 0$, there exists $C = C_{\epsilon, \eta} > 0$ such that*

$$(2.8) \quad \|P(z, h)\varphi^w u\| \geq C_\epsilon h^{2+\eta} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon].$$

3. PROOF OF THEOREM 1 AND COROLLARY 1.3

Recall the conjugated Laplacian is

$$-\tilde{\Delta} = -\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x),$$

where $V_1(x)$ has been computed above. We will do some analysis and reductions now *before* separating variables. If we are considering quasimodes

$$(-\tilde{\Delta} - \lambda^2)u = E(\lambda)\|u\|,$$

where

$$E(\lambda) = \mathcal{O}(\lambda^{-\beta_0})$$

for some $\beta_0 > 0$, then we begin by rescaling. Set $h = \lambda^{-1}$ so that

$$(-h^2\partial_x^2 - h^2A^{-2}(x)\partial_\theta^2 + h^2V_1(x) - 1)u = \tilde{E}(h)\|u\|,$$

where $\tilde{E}(h) = h^2E(h^{-1}) = \mathcal{O}(h^{2+\beta_0})$. With ξ, η the dual variables to x, θ as usual, the semiclassical symbol of this operator is

$$p = \xi^2 + A^{-2}(x)\eta^2 + h^2V_1(x) - 1,$$

and the semiclassical principal symbol is

$$p_0 = \xi^2 + A^{-2}(x)\eta^2 - 1.$$

It is worthwhile to point out that at this point our semiclassical parameter is $h = \lambda^{-1}$. After separating variables later in the proof, we will let $h = |k|^{-1}$, where k is the angular momentum parameter. However, in the regime where we so take $h, |k|$ and λ will be comparable, so it is merely a choice of convenience.

It is important to keep in mind for the remainder of this paper what the various parameters represent. Here, the variable η represents hD_θ . As we will eventually be decomposing in Fourier modes in the θ direction, this means that the variable η takes values in $h\mathbb{Z}$.

We next record that a standard h -parametrix argument tells us that any quasi-mode is concentrated on the energy surface where $\{p_0 = 0\}$. The proof is standard.

Lemma 3.1. *Suppose u satisfies*

$$(-h^2\partial_x^2 - h^2A^{-2}(x)\partial_\theta^2 + h^2V_1(x) - 1)u = \tilde{E}(h)\|u\|,$$

where $\tilde{E}(h) = h^2E(h^{-1}) = \mathcal{O}(h^{2+\beta_0})$, and $\Gamma \in \mathcal{S}^0$ satisfies $\Gamma \equiv 1$ in a small fixed neighbourhood of $\{p_0 = 0\}$. Then

$$(1 - \Gamma^w)u = \mathcal{O}(h^{2+\beta_0}).$$

Hence we will restrict our attention to the characteristic surface where $\{p_0 = 0\}$. Using our moment map idea, we know that η is invariant under the classical flow. Hence if η is very large, our operator will be elliptic, while if η is very small, the parameter ξ will be bounded away from zero, and hence we will have uniform

propagation estimates. Let us make this more precise. Let $A_0 = \min(A(x))$ and $A_1 = \max(A(x))$, and let

$$1 = \psi_0(\eta) + \psi_1(\eta) + \psi_2(\eta)$$

be a partition of unity satisfying

$$\psi_0 \equiv 1 \text{ on } \{|\eta|^2 \leq \frac{1}{2}A_0^2\}$$

with support in $\{|\eta|^2 \leq \frac{3}{4}A_0^2\}$;

$$\psi_2 \equiv 1 \text{ on } \{|\eta|^2 \geq 2A_1^2\}$$

with support in $\{|\eta|^2 \geq \frac{3}{2}A_1^2\}$. Then, on $\text{supp } \psi_0$, we have

$$\eta^2 A^{-2}(x) \leq \eta^2 A_0^{-2} \leq \frac{3}{4},$$

and on $\text{supp } \psi_2$, we have

$$\eta^2 A^{-2}(x) \geq \eta^2 A_1^{-2} \geq \frac{3}{2}.$$

Now for our quasimode u , write

$$u = u_0 + u_1 + u_2 + u_3 := \psi_0^w \Gamma^w u + \psi_1^w \Gamma^w u + \psi_2^w \Gamma^w u + (1 - \Gamma^w)u.$$

Since hD_θ commutes with $-\tilde{\Delta}$ and we can choose $\Gamma = \Gamma(p_0)$ so that $[p_0^w, \Gamma^w] = \mathcal{O}(h^3)$, each of these u_j are also quasimodes of the same order as u (but of course may have small or even trivial L^2 mass).

3.0.6. Estimation of u_0 . Observe that on the support of ψ_0 , since η is invariant, we have $|\xi|^2 \geq 1/4 - \mathcal{O}(h^2)$, which means the propagation speed in the x -direction is bounded below. We claim this implies

$$\|u_0\|_{L^2_{x,\theta}} \leq c_0 \|u_0\|_{L^2([a,b]_x \times \mathbb{S}_\theta)}$$

for some $c_0 > 0$. In other words, u_0 is *uniformly* distributed in the sense that the mass cannot be vanishing in h on any set.

The claim follows by propagation of singularities. The standard propagation of singularities result applies whenever the classical flow propagates singularities from one region to another in phase space. Since we are analyzing the region where $\xi \neq 0$, we have uniform propagation in the x direction. A general statement is given in the following Lemma (a refinement of Hörmander's original result [Hör71]). For a proof in this context, see, for example, [Chr07, Lemma 6.1] and [BZ04, Lemma 4.1].

Lemma 3.2. *Suppose $V_0 \Subset T^*X$, p is a symbol, $T > 0$, A an operator, and $V \Subset T^*X$ a neighbourhood of γ satisfying*

$$(3.1) \quad \begin{cases} \forall \rho \in \{p_0^{-1}(0)\} \setminus V, \exists 0 < t < T \text{ and } \epsilon = \pm 1 \text{ such that} \\ \exp(\epsilon s H_{p_0})(\rho) \subset \{p_0^{-1}(0)\} \setminus V \text{ for } 0 < s < t, \text{ and} \\ \exp(\epsilon t H_{p_0})(\rho) \in V_0; \end{cases}$$

and A is microlocally elliptic in $V_0 \times V_0$. If $B \in \Psi^{0,0}(X, \Omega_X^{\frac{1}{2}})$ and $\text{WF}_h(B) \subset T^*X \setminus V$, then

$$(3.2) \quad \|Bu\| \leq C(h^{-1} \|Pu\| + \|Au\|) + \mathcal{O}(h^\infty) \|u\|.$$

Fix two non-empty intervals in the x direction, (a, b) and (c, d) and assume $u = u_0$ is L^2 normalized. Now using that $Pu = \mathcal{O}(h^{2+\beta_0})\|u\|$, we have

$$\begin{aligned} \|u\|_{L^2((c,d)\times\mathbb{S}^1)} &\leq Ch^{-1}\|Pu\| + C_2\|u\|_{L^2((a,b)\times\mathbb{S}^1)} \\ &\leq Ch^{1+\beta_0}\|u\|_{L^2(\mathbb{S}^1\times\mathbb{S}^1)} + C_2\|u\|_{L^2((a,b)\times\mathbb{S}^1)}, \end{aligned}$$

for some $C_2 > 0$. For $h > 0$ sufficiently small, this implies if u has mass bounded below independent of h in any x neighbourhood (c, d) , then

$$\|u\|_{L^2((a,b)\times\mathbb{S}^1)} \geq c' > 0$$

independent of h . Rescaling in terms of u_0 if u_0 is not normalized, we recover

$$\|u_0\|_{L^2((a,b)\times\mathbb{S}^1)} \geq c'\|u_0\|.$$

Since the interval (a, b) is arbitrary, we have shown that the L^2 -mass on any rotationally invariant neighbourhood is positive independent of h . Thus (1.1) holds with a lower bound independent of $h = \lambda^{-1}$.

3.0.7. Estimation of u_2 . On the other hand, on the support of ψ_2 , we have the principal symbol satisfies

$$|p_0| \geq \frac{1}{2},$$

so we claim that an elliptic argument shows

$$\|u_2\|_{L^2} = \mathcal{O}(h^\infty)\|u_2\|_{L^2}.$$

That is, since $|p_0| \geq \frac{1}{2}$ on support of ψ_2 , there is an h -parametrix for P there: there exists Q such that

$$QP\psi_2^w = \psi_2^w + \mathcal{O}(h^\infty),$$

and further Q has bounded L^2 norm. Hence

$$\begin{aligned} \|u_2\| &= \|QPu_2\| + \mathcal{O}(h^\infty)\|u_2\| \\ &\leq C\|Pu_2\| + \mathcal{O}(h^\infty)\|u_2\| \\ &= \mathcal{O}(h^{2+\beta_0})\|u_2\|. \end{aligned}$$

This implies $u_2 = \mathcal{O}(h^\infty)$.

3.0.8. Estimation of u_1 . In order to consider the final part u_1 , which is microsupported where all the critical points are, we will employ one further reduction. Since u_1 is microsupported in a region where $|\eta|$ is bounded between two constants, say, $a_0 \leq |\eta| \leq a_1$, and $\eta = hk$ for some integer k , a priori the number of angular momenta k in the wavefront set of u_1 is comparable to h^{-1} . We can do better than that. Using the semiclassical calculus, we will next show that there exists $k_0 \in \mathbb{Z}$ such that for any $\epsilon > 0$, we have

$$u_1 = \sum_{|k-k_0| \leq h^{-\epsilon}} e^{ik\theta} \varphi_k(x) + \mathcal{O}(h^\infty)\|u_1\|.$$

That is, we claim that the Fourier decomposition of u_1 can actually only have $\mathcal{O}(h^{-\epsilon})$ non-trivial modes. To prove this claim, fix $k_0 \in \mathbb{Z}$ and any $\epsilon > 0$, and choose a $k_1 \in \mathbb{Z}$ satisfying

$$|k_1 - k_0| \geq h^{-\epsilon}.$$

We will show that we can decompose u_1 into (at least) two pieces with disjoint microsupport, one near hk_0 and one near hk_1 . Evidently, these two pieces correspond to different angular momenta η , so have wavefront sets associated to different level

sets of the moment map. Of course, level sets sufficiently close (in an h -dependent set) may contribute to a single irreducible quasimode, but the point is to quantify how far away from a single level set one needs to go before leaving the microsupport of an irreducible quasimode.

In order to make this rigorous, let $\eta_j = hk_j$ for $j = 0, 1$, and choose $\chi(r) \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfying

$$\chi(r) \equiv 1 \text{ for } |r| \leq 1,$$

with support in $\{|r| \leq 2\}$. For $j = 0, 1$, let

$$\chi_j(\eta, h) = \chi\left(\frac{\eta - \eta_j}{h^{1-\epsilon/2}}\right).$$

As semiclassical symbols, the χ_j are in a harmless $h^{1/2-\epsilon/4}$ calculus, and moreover they only depend on η (not on θ) and commute with $-\tilde{\Delta}$. On the support of each of the χ_j , we have

$$\left|\frac{\eta - \eta_j}{h^{1-\epsilon/2}}\right| = \left|\frac{hk - hk_j}{h^{1-\epsilon/2}}\right| = \left|\frac{k - k_j}{h^{-\epsilon/2}}\right| \leq 2.$$

This implies

$$|k - k_j| \leq 2h^{-\epsilon/2},$$

so as $h \rightarrow 0+$, χ_0 and χ_1 have disjoint supports. This means the functions $\chi_1 u_1$ and $\chi_2 u_1$ have disjoint h -wavefront sets, so they are almost orthogonal:

$$\langle \chi_1 u_1, \chi_2 u_1 \rangle = \mathcal{O}(h^\infty).$$

Hence if each of these functions has nontrivial L^2 mass, then u was not an irreducible quasimode.

Finally, we analyse the function u_1 , but spread over at most $\mathcal{O}(h^{-\epsilon})$ Fourier modes. Throughout the remainder of this section, let λ be large and fixed. Let us consider a single Fourier mode confined to a single angular momentum k .

The case of u_1 is the most interesting case, as the microsupport of u_1 contains all of the critical elements. Now recalling again the separated equation (2.1) with the potential

$$V(x) = A^{-2}(x) + h^2 V_1(x),$$

let A_0 and A_1 again be the min/max respectively of $A(x)$. Our spectral parameter now is $z = h^2 \lambda^2$. We are localized where

$$\frac{1}{2} A_0^2 \leq (\lambda^{-1} k)^2 \leq 2 A_1^2,$$

or

$$\frac{1}{2} A_1^{-2} \leq z \leq 2 A_0^{-2}.$$

This of course implies that λ and k are comparable. Let $(a, b) \subset \mathbb{S}^1$ be a non-empty interval. We need to show that if u is a weak irreducible quasimode,

$$((hD)^2 + V(x) - z)u = \mathcal{O}(h^{2+\beta_0})\|u\|,$$

with $\|u\| = 1$, then either $\|u\|_{L^2(a,b)} = \mathcal{O}(h^\infty) = \mathcal{O}(\lambda^{-\infty})$, or $\|u\|_{L^2(a,b)} \geq C_\epsilon h^{1+\epsilon}$ for any $\epsilon > 0$. Let us assume that u is nontrivial so that $\|u\|_{L^2(a,b)} \geq ch^N$ for some N .

There are a number of subcases to consider here. We observe that, according to Lemma 3.2, we can always microlocalize further to a set close to the energy level

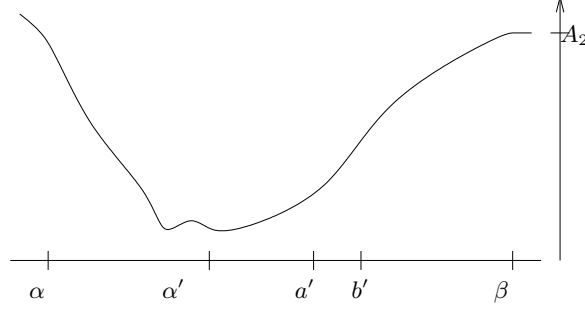


FIGURE 3. The function $A^{-2}(x)$ and the weakly unstable critical point β .

of interest. That is, for $P(z, h) = (hD)^2 + V(x) - z$, if $P(z, h)u = \mathcal{O}(h^{2+\beta_0})$, then if $\psi(r) \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies $\psi \equiv 1$ for r near 0, we have for any $\delta > 0$

$$\psi^w((\xi^2 + V(x) - z)/\delta)u = u + \mathcal{O}(h^{2+\beta_0}).$$

For the rest of this section, we write ψ^w for this energy cutoff.

Case 1:

Next, assume z is in a small neighbourhood of a critical energy level, and assume $A'(x) \neq 0$ somewhere on (a, b) . Then let (a', b') be a non-empty interval with

$$(a', b') \Subset \{A' \neq 0\} \cap (a, b),$$

and let $(\alpha', \beta) \supset (a', b')$ be the maximal connected interval with $A'(x) \neq 0$ on (α', β) . Now A' has constant sign on (α', β) , so at least one of α' or β is part of a weakly unstable critical element (see Figure 3).

Without loss in generality, assume $A' < 0$ on (α', β) so that at least β lies in a weakly unstable critical element. That is, the principal part of the potential $A^{-2}(x)$ increases as $x \rightarrow \beta^-$, and takes the value, say $A^{-2}(\beta) = A_2$. Let (α, β) be the maximal open interval containing (α', β) where $A^{-2}(x) < A_2$ on (α, β) . As $A^{-2}(x) < A^{-2}(\alpha)$ for $x \in (\alpha, \beta)$ and $A^{-2}(\alpha) = A_2$, we have $(A^{-2}(x))' < 0$ for $x \in (\alpha, \beta)$ sufficiently close to α . That means that either α is part of a weakly unstable critical element, or $A'(\alpha) \neq 0$. We break the analysis into the two separate subsubcases, beginning with $A'(\alpha) \neq 0$.

Case 1a: If $A'(\alpha) \neq 0$, then the weakly unstable/stable manifolds associated to $(A^{-2})'(\beta) = 0$ are homoclinic to each other (see Figure 4), and in particular, propagation of singularities can be used to control the mass along this whole trajectory, as long as we stay away from the right hand endpoint β . That is, propagation of singularities implies for any $\eta > 0$ independent of h ,

$$\begin{aligned} \|\psi^w u\|_{L^2(\alpha, \beta - \eta)} &\leq C_\eta (h^{-1} \|((hD)^2 + V - z)\psi^w u\| + \|\psi^w u\|_{L^2(a', b')}) \\ &\leq C_\eta h^{1+\beta_0} \|u\| + \|\psi^w u\|_{L^2(a', b')}. \end{aligned}$$

Hence by taking $h > 0$ sufficiently small, we need to bound $\|\psi^w u\|_{L^2(\alpha, \beta - \eta)}$ from below in terms of $\|u\|$.

Let $[\beta, \kappa]$ be the maximal connected interval containing β on which $A' = 0$ (we allow $\kappa = \beta$ if the critical point is isolated). Let $\tilde{\chi} \equiv 1$ on $[\beta, \kappa]$ with support in a small neighbourhood thereof, and let $\chi \equiv 1$ on $\text{supp } \tilde{\chi}$ with support in a

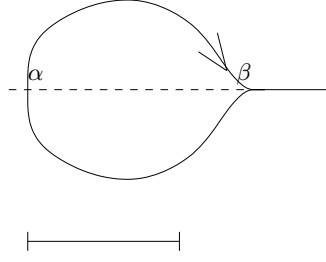


FIGURE 4. If $A'(\alpha) \neq 0$, the unstable manifold from β flows into the stable manifold at β (homoclinicity). The interval indicates a region with propagation speed uniformly bounded below.

slightly smaller set so that $(1 - \tilde{\chi}) \geq (1 - \chi)$ and $(1 - \tilde{\chi}) \geq c|\chi'|$. Then writing $P(z, h) = (hD)^2 + V - z$, we have from Theorem 2 (for any $\epsilon > 0$)

$$\begin{aligned} \|u\| &\leq \|\chi u\| + \|(1 - \chi)u\| \\ &\leq C_\epsilon h^{-2-\epsilon} \|P(z, h)\chi u\| + \|(1 - \tilde{\chi})u\| \\ &\leq C_\epsilon h^{-2-\epsilon} (\|\chi P(z, h)u\| + \|[P(z, h), \chi]u\|) + \|(1 - \tilde{\chi})u\| \\ &\leq C'_\epsilon (h^{\beta_0 - \epsilon} \|u\| + h^{-1-\epsilon} \|(1 - \tilde{\chi})u\|) + \|(1 - \tilde{\chi})u\| \end{aligned}$$

Rearranging and taking $h > 0$ sufficiently small and $\epsilon < \beta_0$, we get

$$(3.3) \quad \|(1 - \tilde{\chi})u\| \geq C_\epsilon h^{1+\epsilon} \|u\|.$$

Now either the wavefront set of u is contained in the closure of the lift of (α, β) or it isn't. In the latter case there is nothing to prove. In the former case, we conclude that $u = \mathcal{O}(h^\infty)$ on any open subset whose closure does not meet the set $[\alpha, \beta]$. We appeal to propagation of singularities one more time. Since $A'(\alpha) \neq 0$, propagation of singularities applies in a neighbourhood of α , so that (shrinking $\eta > 0$ if necessary) for some $c_1 > 0$,

$$\|u\|_{L^2(\alpha, \beta - \eta)} \geq c_1 \|u\|_{L^2(\alpha - \eta, \beta - \eta)}.$$

Since we have assumed $u = \mathcal{O}(h^\infty)$ on $(\alpha - \eta, \beta + \eta)^c$, this estimate, together with (3.3) and (3.2) allows us to conclude

$$\|u\|_{L^2(\alpha, \beta - \eta)} \geq C \|(1 - \tilde{\chi})u\| \geq C_\epsilon h^{1+\epsilon} \|u\|.$$

Case 1b: We now consider the possibility that $A'(\alpha) = 0$ as well as $A'(\beta) = 0$ (see Figure 5). In this case, propagation of singularities fails at both endpoints of (α, β) , so we can only conclude that for any $\eta > 0$ independent of h ,

$$\|u\|_{L^2(\alpha + \eta, \beta - \eta)} \leq C_\eta (h^{-1} \|((hD)^2 + V - z)u\| + \|u\|_{L^2(\alpha', \beta')}).$$

Hence now it suffices to prove that for some $\eta > 0$ small but independent of h , we have the estimate

$$\|u\|_{L^2(\alpha + \eta, \beta - \eta)} \geq C_\epsilon h^{1+\epsilon} \|u\|$$

for any $\epsilon > 0$.

Let $[\beta, \kappa]$ be the maximal connected interval containing β on which $A' = 0$, and let $[\omega, \alpha]$ be the maximal connected interval containing α on which $A' = 0$. Let $\tilde{\chi} \equiv 1$ on $[\beta, \kappa] \cup [\omega, \alpha]$ with support in small neighbourhoods thereof, and let $\chi \equiv 1$

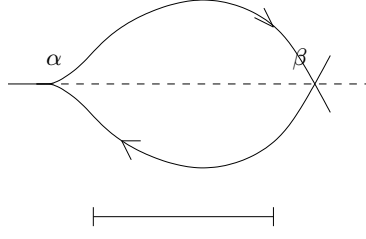


FIGURE 5. If $A'(\alpha) = 0$, the unstable manifold from β flows into the stable manifold at α and vice versa. The interval indicates a region with propagation speed uniformly bounded below.

on $\text{supp } \tilde{\chi}$ with support in a slightly smaller set so that $(1 - \tilde{\chi}) \geq (1 - \chi)$ and $(1 - \tilde{\chi}) \geq c|\chi'|$. Since both $[\omega, \alpha]$ and $[\beta, \kappa]$ are weakly unstable, we can apply Theorem 2 and the same argument as above to finish this case.

Case 2: Finally, we assume $(a, b) \subset \{A' = 0\}$. Again, if $A^{-2} \equiv A_3$ on (a, b) and $z \neq A_3$, we can use propagation of singularities to control $\|u\|_{L^2(a,b)}$ from below by its mass on the connected component in $\{p = z\}$ containing (a, b) (as in the case of u_0 above). Hence we are interested in the case where z is in a small neighbourhood of A_3 .

If $u = \mathcal{O}(h^\infty)$ on (a, b) there is nothing to prove, so assume not. Then if $[\alpha, \beta] \supset (a, b)$ is the maximal connected interval where $A^{-2}(x) \equiv A_3$, the wavefront set of u is contained in a small neighbourhood of $[\alpha, \beta]$, so that for $\delta > 0$ as small as we like by taking a sufficiently localized energy cutoff, we have

$$\|u\|_{L^2([\alpha-\delta, \beta+\delta]^\epsilon)} = \mathcal{O}_\delta(h^\infty).$$

That means that, either

$$\|u\|_{L^2([a,b])} \geq c > 0, \quad \|u\|_{L^2([\alpha-\delta, a])} \geq c > 0, \quad \text{or} \quad \|u\|_{L^2([b, \beta+\delta])} \geq c > 0.$$

If the first estimate is true, we're done, so assume without loss in generality that $\|u\|_{L^2([b, \beta+\delta])} \geq c > 0$. Assume for contradiction that there exists $\epsilon_0 > 0$ such that $\|u\|_{L^2(a,b)} \leq Ch^{1+\epsilon_0}$. Let $\chi \in \mathcal{C}_c^\infty$ be a smooth function such that $\chi \equiv 1$ on $[b, \beta + \delta]$ with support in $(a, \beta + 2\delta)$. Write $\tilde{u} = \chi u$. If $[\alpha, \beta]$ is a weakly stable critical element, modify $A^{-2}(x)$ on the support of $1 - \chi$ so that $[\alpha, \beta]$ is weakly unstable. That is, if $(A^{-2}(x))' < 0$ for $x < \alpha$ in some neighbourhood, replace A with a locally defined function \tilde{A} satisfying $\tilde{A} \equiv A$ on $\text{supp } \chi$ but $(\tilde{A}^{-2}(x))' > 0$ for $x < \alpha$ in some neighbourhood. If $[\alpha, \beta]$ is weakly unstable, then let $\tilde{A} \equiv A$ (see Figure 6). We apply Theorem 2 once again (for any $\epsilon > 0$):

$$\begin{aligned} \|\chi u\| &\leq C_\epsilon h^{-2-\epsilon} \|((hD)^2 + \tilde{A}^{-2} + h^2 V_1 - z)\chi u\| \\ &= C_\epsilon h^{-2-\epsilon} \|((hD)^2 + A^{-2} + h^2 V_1 - z)\chi u\| \\ &\leq C_\epsilon h^{-2-\epsilon} (\|P(z, h)u\| + \|[P(z, h), \chi]u\|) \\ &\leq C'_\epsilon (h^{\epsilon_0-\epsilon} \|u\| + h^{-1-\epsilon} \|u\|_{L^2(a,b)}) + \mathcal{O}(h^\infty), \end{aligned}$$

where the $\mathcal{O}(h^\infty)$ error comes from the part of the commutator $[P(z, h), \chi]$ supported outside a neighbourhood of $[\alpha, \beta]$ (the other part contributing the integral over (a, b)). But our contradiction assumption implies that the right hand side is $o(1)$ as $h \rightarrow 0$ provided $\epsilon < \epsilon_0$. As $\|\chi u\| \geq c > 0$, this is a contradiction.

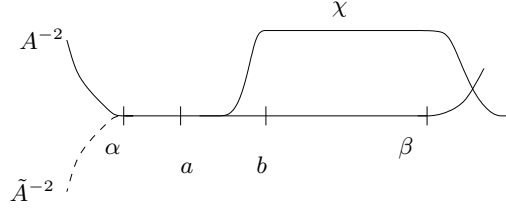


FIGURE 6. The setup for Case 2. Here if the quasimode is small in (a, b) , we cut off to the right of (a, b) and modify A^{-2} to the left to be weakly unstable. We then arrive at a contradiction.

3.1. Finishing up the proof. We now put together the estimates of u_0, u_1, u_2, u_3 . Since $u_3 = \mathcal{O}(h^{2+\beta_0})$ and $u_2 = \mathcal{O}(h^\infty)$, for $h > 0$ sufficiently small, at least one of u_0 and u_1 must have L^2 mass bounded below independent of h . If u_0 has L^2 mass bounded below independent of h we're done by the propagation of singularities argument in Subsection 3.0.6. Hence we need to conclude Theorem 1 assuming u_0 is small and u_1 carries most of the L^2 mass.

Fix (a, b) as considered in Subsection 3.0.8 and recall we know that for any $\epsilon > 0$

$$u_1 = \sum_{|k-k_0| \leq h^{-\epsilon}} e^{ik\theta} \varphi_k(x) + \mathcal{O}(h^\infty) \|u_1\|.$$

We use the notation $\Omega = (a, b)_x \times \mathbb{S}_\theta^1$ as in the statement of Theorem 1. Each φ_k satisfies either

$$\|\varphi_k\|_{L^2(a,b)} = \mathcal{O}(|k|^{-\infty})$$

or for any $\epsilon > 0$,

$$\|\varphi_k\|_{L^2(a,b)} \geq c_2 |k|^{-1-\epsilon} \|\varphi_k\|_{L^2(\mathbb{S}_x^1)}.$$

In the first case, these φ_k s have disjoint wavefront sets from the φ_k s in the latter case, so leaving them in the sum would mean our quasimode was not irreducible. Removing these from the sum and reindexing if necessary, we conclude

$$\begin{aligned} \|u_1\|_{L^2(\Omega)}^2 &= \sum_{|k-k_0| \leq k_0^\epsilon} \|\varphi_k(x)\|_{L^2(a,b)}^2 + \mathcal{O}(h^\infty) \|u_1\|^2 \\ &\geq c_2' k_0^{-2-2\epsilon} \sum_{|k-k_0| \leq k_0^\epsilon} \|\varphi_k(x)\|_{L^2(\mathbb{S}_x^1)}^2 - \mathcal{O}(h^\infty) \|u_1\|^2 \\ &= c_2'' \lambda^{-2-2\epsilon} \|u_1\|_{L^2(\mathbb{S}_x^1 \times \mathbb{S}_\theta^1)}^2 - \mathcal{O}(\lambda^{-\infty}) \|u_1\|^2. \end{aligned}$$

This concludes the proof of Theorem 1. □

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