SHARP LOCAL SMOOTHING FOR WARPED PRODUCT MANIFOLDS WITH SMOOTH INFLECTION TRANSMISSION

HANS CHRISTIANSON AND JASON METCALFE

ABSTRACT. We consider a family of rotationally symmetric, asymptotically Euclidean manifolds with two trapped sets, one which is unstable and one which is semi-stable. We prove a sharp local smoothing estimate for the linear Schrödinger equation with a loss which depends on how flat the manifold is near each of the trapped sets. The result interpolates between the family of similar estimates in [CW11]. As a consequence of the techniques of proof, we also show a sharp high energy resolvent estimate with a polynomial loss depending on how flat the manifold is near each of the trapped sets.

1. INTRODUCTION

In this paper we study the local smoothing effect for the Schrödinger equation on a class of warped product manifolds with a trapped set that is mixed unstable and *semistable*, which we call inflection-transmission due to its relation to classical transmission problems (see Remark 1.4). Our main result is a generalization of the local smoothing estimate

$$\int_0^T \|\langle x\rangle^{-1/2-} e^{it\Delta} u_0\|_{H^{1/2}}^2 \, dt \lesssim \|u_0\|_{L^2}^2.$$

Such estimates first appeared in [CS88], [Sjö87], [Veg98] and were extended to non-trapping asymptotically flat geometries in [CKS95], [Doi96a]. See, e.g., [RT07], [MMT08] for some recent generalizations. The presence of trapping necessitates a loss of smoothing as was shown in [Doi96b]. If the trapping is unstable and nondegenerate, this has already been studied in [Bur04], [Chr07, Chr08, Chr11], [Dat09], [BGH10] amongst several others. Trapping that is unstable but degenerately so was the topic of [CW11]. The novel thing in this paper is the existence of semistable trapping, that is, trapping which is stable from one direction and unstable from another direction.

Let us begin by describing the geometry. We shall demonstrate the effect of this sort of trapping by examining an explicit example. Let m_1 and m_2 be positive integers, and set

$$A^2(x) = 1 + \int_0^x y^{2m_1-1}(y-1)^{2m_2}/(1+y^2)^{m_1+m_2-1} dy.$$

As the integrand in the last term

$$x^{2m_1-1}(x-1)^{2m_2}/(1+x^2)^{m_1+m_2-1} \sim \begin{cases} x^{2m_1-1}, & x \sim 0, \\ (x-1)^{2m_2}/2^{m_1+m_2-1} & x \sim 1, \\ x, & |x| \to \infty, \end{cases}$$

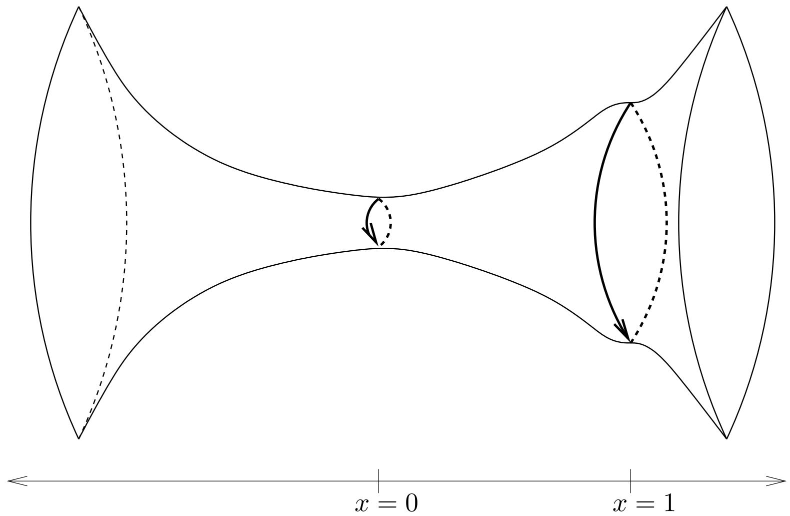

FIGURE 1. A piece of the manifold X with the trapped sets at $x = 0$ and at $x = 1$.

we notice that

$$A^2(x) \sim \begin{cases} 1 + x^{2m_1}, & x \sim 0, \\ C_1 + c_2(x-1)^{2m_2+1} & x \sim 1, \\ x^2, & |x| \to \infty. \end{cases} \tag{1.1}$$

Here $C_1 > 1$ and $c_2 < 1$ are constants which are easily computed but inessential, except for their relative sizes compared to 1. As will be clear in the sequel, the specific structure of A is inessential and only the location and nature of the critical points and behavior at infinity matter.

Now let $X = \mathbb{R}_x \times \mathbb{R}_\theta / 2\pi\mathbb{Z}$, equipped with the metric

$$ds^2 = dx^2 + A^2(x) d\theta^2,$$

so that X is asymptotically Euclidean with two ends and has two trapped sets. The trapping occurs where $A'(x) = 0$, which is at $x = 0$ and $x = 1$ respectively (see Figure 1). The metric determines the volume form

$$d\text{Vol} = A(x) dx d\theta$$

and the Laplace-Beltrami operator acting on 0-forms

$$\Delta f = (\partial_x^2 + A^{-2}\partial_\theta^2 + A^{-1}A'\partial_x) f.$$

More general warped product manifolds (with arbitrary compact cross section) can also be considered (see Remark 1.3 and [Chr13]), however for simplicity in exposition, we consider only the surface of revolution case here.

Our main result is the following local smoothing estimate with sharp loss. Using the common notation $D_t = (1/i)\partial_t$, we have:

Theorem 1 (Local Smoothing)**.** *Suppose X is as above with $m_1, m_2 \geqslant 1$ and assume u solves*

$$\begin{cases} (D_t - \Delta)u = 0 \text{ in } \mathbb{R} \times X, \\ u|_{t=0} = u_0 \in H^s \end{cases}$$

for some $s > 0$ sufficiently large. Then for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\int_0^T \left(\| \langle x \rangle^{-1} \partial_x u \|^2_{L^2(d\,Vol)} + \| \langle x \rangle^{-3/2} \partial_\theta u \|^2_{L^2(d\,Vol)} \right) dt$$
$$\leqslant C_T \left(\| \langle D_\theta \rangle^{\beta(m_1,m_2)} u_0 \|^2_{L^2(d\,Vol)} + \| \langle D_x \rangle^{1/2} u_0 \|^2_{L^2(d\,Vol)} \right),$$

where

$$\beta(m_1, m_2) = \max \left(\frac{m_1}{m_1+1}, \frac{2m_2+1}{2m_2+3} \right). \tag{1.2}$$

Moreover this estimate is sharp, in the sense that no polynomial improvement in regularity is true.

This theorem requires some remarks.

Remark 1.1. Observe that the maximum *gain* in regularity in the presence of inflection-transmission trapping is $2/(2m_2+3)$ derivatives. Each of these fractions lies in between sequential fractions in the numerology of [CW11], since

$$\frac{1}{(m+1)+1} < \frac{2}{2m+3} < \frac{1}{m+1}.$$

Remark 1.2. In the theorem above, the weights at infinity are different than those that appear in the standard Euclidean estimate. Standard cutoff arguments would allow us to make these match. The key new aspect of the theorem, however, is the behavior near the trapped sets, and for clarity in the proof, we do not modify the weights.

Remark 1.3. Theorem 1 and indeed also Theorem 2 below are of course true in many more situations. In our example chosen to illustrate the effect of such trapping, we are working on a surface of revolution where the generating curve is parametrized by arclength. The techniques herein apply more generally to warped-product manifolds. In particular, $\mathbb{R}_\theta / 2\pi\mathbb{Z}$ maybe replaced by a compact manifold of any dimension. The inflection-transmission trapping is characterized by an inflection point in the potential that results from a reduction to a one dimensional problem via an eigenvalue decomposition of the compact portion of the manifold.

Of particular interest, the microlocalization step which separates the trapped sets at different energies used to prove (2.8) indicates the same result applies to a manifold with one Euclidean end and only an inflection transmission trapped set. On the other hand, if our manifold has two Euclidean ends, a degenerate hyperbolic trapped set, and *two* inflection transmission trapped sets *at the same* semiclassical energy, it is natural to suspect that such a theorem is no longer true because the two inflection transmission sets must tunnel to each other. However, it is easy to see that the theorem still applies in this case, since the stable/unstable manifolds for the degenerate hyperbolic trapped set form a separatrix (in other words, the degenerate hyperbolic trapped set is at *higher* semiclassical energy). Hence the same microlocalization applies, and so does the theorem.

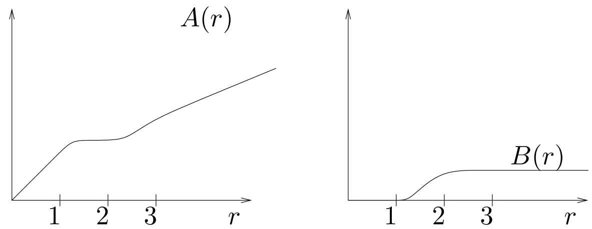

FIGURE 2. The functions $A(x)$ and $B(x)$.

Remark 1.4. We briefly discuss why we have chosen to call this type of smooth trapping "inflection-transmission" type trapping. The inflection part refers to the fact that the effective potential after separating variables has an inflection point at the trapped set. We have also included transmission in our name because this kind of trapping bears some resemblance to the traditional transmission problem.

The traditional transmission problem concerns a wave equation in a medium for which the speed of propagation is distinct in different regions. For example, one might study solutions to the equation

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 \text{ for } |x| < 1, \\ (\partial_t^2 - c^2\Delta)u = 0 \text{ for } |x| > 1, \end{cases}$$

where $c \neq 1$. Of course one also needs to indicate appropriate boundary conditions at the interaction surface where $|x| = 1$ (see, for example, [CPV01, CPV99, CV10]).

On the other hand, if we consider a surface of revolution given by a specific generating curve C in the (x_1, x_3) plane, rotated around the x_3 axis, we get a similar looking picture. Let

$$C = \{(x_1, 0, x_3) = (A(r), 0, B(r)), r \geqslant 0\},$$

where

$$A(r) = \begin{cases} r, \text{ for } 0 \leqslant r \leqslant 1, \\ \frac{1}{2}r, \text{ for } r \geqslant 3, \end{cases}$$

and assume $0 \leqslant A'(r) \leqslant 1$ and A has an inflection point at, say, $r = 2$. The function $A(r)$ is sketched schematically in Figure 2.

We suppose that B' is compactly supported in the region $1 \leq r \leq 3$ and fix $B(0) = 0$. The function $B(r)$ is also depicted in Figure 2.

Rotating the curve C about the x_3 axis in $\mathbb{R}^3$ yields a manifold which is flat near 0 and flat outside a compact set, and changes "height" in between (see Figure 3).

Moreover, if we compute the Laplacian on this surface of revolution, we see that

$$\Delta_g = \frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_\theta^2, \quad 0 \leq r \leq 1$$

but

$$\Delta_g = 4\Big(\frac{1}{r}\partial_r r \partial_r + \frac{1}{r^2}\partial_\theta^2\Big), \quad r \geq 3.$$

See, e.g., [Boo11] where such a computation has been carried out in much detail.

As in [CW11], once we prove Theorem 1, we can obtain a resolvent bound. For simplicity, say that our surface of revolution is Euclidean at infinity. That is,

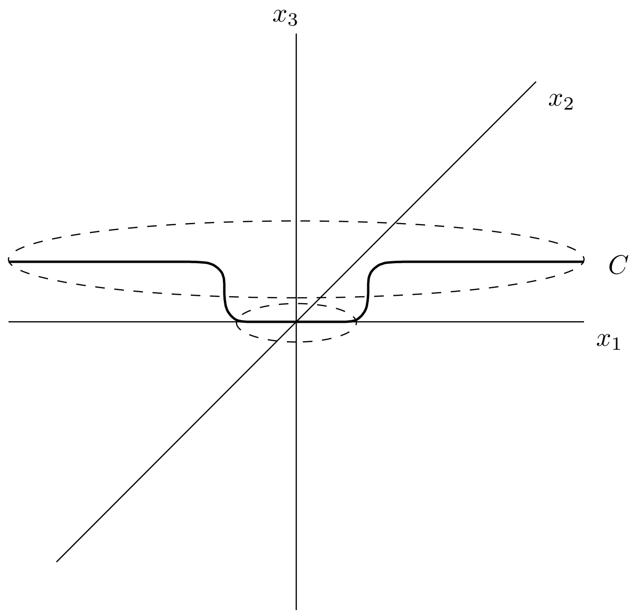

FIGURE 3. The manifold obtained by rotating the curve C about the x_3 axis.

assume $A(x) = x$ for $|x| \gg 0$. Alternatively, we could require dilation analyticity at infinity, which would permit asymptotically conic spaces as were treated in [WZ00].

We let

$$R(\lambda) = (-\Delta_g - \lambda^2)^{-1}$$

denote the resolvent on X (where it exists), and take $\operatorname{Im} \lambda < 0$ as our physical sheet. With a choice of appropriate branch cut, $\chi R(\lambda)\chi$ extends meromorphically to $\{\lambda \in \mathbb{R} : \lambda \gg 0\}$ for any $\chi \in C_c^\infty(X)$. See, e.g., [SZ91]. And, in the degenerate inflection point setting, we have

Theorem 2. *For any $\chi \in \mathcal{C}_c^\infty(X)$, there exists a constant $C = C_{m_1,m_2,\chi} > 0$ such that for $\lambda \gg 0$,*

$$\|\chi R(\lambda - i0)\chi\|_{L^2 \to L^2} \leqslant C \max\{\lambda^{-2/(m_1+1)}, \lambda^{-4/(2m_2+3)}\}.$$

Moreover, this is the nut estimate, in the sense that no better polynomial rate of decay holds true for every cutoff χ.

Acknowledgements. The first author was supported, in part, by NSF grant DMS-0900524, and the second author by NSF grant DMS-1054289.

2. Local smoothing estimates

In this section, we shall prove the main local smoothing estimate. In the subsequent section, we shall saturate the inequality, thus showing that the loss is sharp. The proof of the estimate uses a positive commutator argument. On Euclidean space, such a proof of local smoothing is well-known, though the interested reader can see [CW11, Section 2.1] for an exposition which is quite akin to what follows.

We first conjugate Δ and reduce to a one dimensional problem. Indeed, we set $L : L^2(X, d\mathrm{Vol}) \to L^2(X, dx d\theta)$ to be the isometry

$$Lu(x,\theta) = A^{1/2}(x)u(x,\theta).$$

With mild assumptions on A, $\widetilde{\Delta} = L\Delta L^{-1}$ is (essentially) self-adjoint on $L^2(X, dxd\theta)$. More explicitly, we have

$$-\widetilde{\Delta} f = (-\partial_x^2 - A^{-2}(x)\partial_\theta^2 + V_1(x))f$$

with

$$V_1(x) = \frac{1}{2}A'' A^{-1} - \frac{1}{4}(A')^2 A^{-2}.$$

Given a function ψ on X, we expand into its Fourier series, $\psi(x,\theta) = \sum_k \varphi_k(x)e^{ik\theta}$, and note that

$$(-\widetilde{\Delta} - \lambda^2)\psi = \sum_k e^{ik\theta}(P_k - \lambda^2)\varphi_k(x),$$

where

$$P_k\varphi_k(x) = \Big(-\frac{d^2}{dx^2} + k^2 A^{-2}(x) + V_1(x)\Big)\varphi_k(x).$$

By setting $h = k^{-1}$, we pass to the semiclassical operator

$$(P(h) - z)\varphi(x) = \Big(-h^2\frac{d^2}{dx^2} + V(x) - z\Big)\varphi(x),$$

where the potential is

$$V(x) = A^{-2}(x) + h^2 V_1(x)$$

and the spectral parameter is $z = h^2\lambda^2$.

We first show

Proposition 2.1. *Suppose u solves*

$$\begin{cases} (D_t - \widetilde{\Delta})u = 0, \\ u(0,x,\theta) = u_0. \end{cases} \tag{2.1}$$

Then for any $T < \infty$, there exists a constant $C_T > 0$ such that

$$\int_0^T \Big(\| \langle x\rangle^{-1} \partial_x u\|^2_{L^2(dx\,d\theta)} + \| \langle x\rangle^{-3/2} \partial_\theta u\|^2_{L^2(dx\,d\theta)} \Big)\, dt \\ \leqslant C_T \Big(\| \langle D_\theta\rangle^{\beta(m_1,m_2)} u_0\|^2_{L^2(dx\,d\theta)} + \| \langle D_x\rangle^{1/2} u_0\|^2_{L^2(dx\,d\theta)} \Big),$$

where $\beta(m_1, m_2)$ is as in (1.2).

The equation (2.1) is obtained by conjugating the original equation by the operator L. Upon conjugating back, Proposition 2.1 shows that the estimate of Theorem 1 holds.

2.1. **Proof of Proposition 2.1.** The proof will be broken into three steps. The first is to use a positive commutator argument to prove full smoothing away from the periodic orbits at $x = 0$ and $x = 1$. We then expand into a Fourier series to reduce to a one dimensional problem, and we reduce the problem to understanding the high frequency part. Using a TT^* argument, gluing techniques and a semiclassical rescaling, we show that the high frequency estimate follows from a cutoff resolvent estimate near each instance of trapping and subsequently prove those.

2.1.1. *The estimate away from $x = 0$ and $x = 1$.* For a self-adjoint operator $\tilde{\Delta}$ and a time-independent, self-adjoint multiplier B, we have

$$\frac{d}{dt}\langle u, Bu\rangle = -2\operatorname{Im}\langle (D_t - \tilde{\Delta})u, Bu\rangle + i\langle[-\tilde{\Delta}, B]u, u\rangle. \tag{2.2}$$

In particular, if

$$B = \frac{1}{2}\arctan(x)D_x + \frac{1}{2}D_x\arctan(x),$$

then

$$i[-\widetilde{\Delta}, B] = 2D_x\langle x\rangle^{-2}D_x + 2D_\theta A' A^{-3}\arctan(x)D_\theta - \frac{3x^2-1}{\langle x\rangle^6} - V_1'\arctan(x).$$

Upon integrating (2.2) over $[0, T]$, we obtain

$$\int_0^T \langle i[-\widetilde{\Delta}, B]u, u\rangle\, dt = i\langle u, \arctan(x)\partial_x u\rangle\Big|_0^T - \frac{i}{2}\langle u, \langle x\rangle^{-1}u\rangle\Big|_0^T$$

for a solution u to (2.1). Using energy estimates, the right side is controlled by $\|u_0\|^2_{H^{1/2}}$. Noting also that energy estimates permit the control

$$\Big|\int_0^T \Big\langle \frac{3x^2-1}{\langle x\rangle^6}u + V_1'\arctan(x)u, u\Big\rangle\, dt\Big| \leqslant CT\sup_{0\le t\le T}\|u(t,\cdot)\|^2_{L^2} \leqslant C_T\|u_0\|^2_{H^{1/2}},$$

it now follows from integration by parts that we have established

$$\int_0^T \Big(\|\langle x\rangle^{-1}\partial_x u\|^2_{L^2} + \|\sqrt{A'A^{-3}\arctan(x)}\partial_\theta u\|^2_{L^2}\Big)\, dt \leqslant C_T\|u_0\|^2_{H^{1/2}}.$$

We observe that

$$A'A^{-3}\arctan(x) \geqslant 0$$

and satisfies

$$A'A^{-3}\arctan(x) \sim \begin{cases} x^{2m_1}, & x\sim 0,\\ c_2'(x-1)^{2m_2}, & x\sim 1,\\ |x|^{-3}, & |x|\to\infty.\end{cases}$$

Thus,

$$\||x|^{m_1}|x-1|^{m_2}\langle x\rangle^{-m_1-m_2-3/2}\partial_\theta u\|_{L^2} \leqslant C\|\sqrt{A'A^{-3}\arctan(x)}\partial_\theta u\|_{L^2},$$

and hence we have the estimate

$$\int_0^T \Big(\|\langle x\rangle^{-1}\partial_x u\|^2_{L^2} + \||x|^{m_1}|x-1|^{m_2}\langle x\rangle^{-m_1-m_2-3/2}\partial_\theta u\|^2_{L^2}\Big)\, dt \leqslant C_T\|u_0\|^2_{H^{1/2}}. \tag{2.3}$$

That is, we have perfect smoothing in the radial direction and in the θ direction away from $x = 0$ and $x = 1$, which is precisely where the trapped sets reside.

2.1.2. *Fourier decomposition.* To get an estimate in the directions tangential to the trapping, we decompose into Fourier series

$$u(t, x, \theta) = \sum_k e^{ik\theta}u_k(t, x)$$

and

$$u_0(x, \theta) = \sum_k e^{ik\theta}u_{0,k}(x).$$

By Plancherel's theorem, it suffices to show

$$\int_0^T \Big(\| \langle x\rangle^{-1} \partial_x u_k\|^2_{L^2(dx)} + k^2 \| \langle x\rangle^{-3/2} u_k\|^2_{L^2(dx)}\Big)\,dt \leqslant C_T\Big(\| \langle k\rangle^{\beta(m_1,m_2)} u_{0,k}\|^2_{L^2(dx)} + \| \langle D_x\rangle^{1/2} u_{0,k}\|^2_{L^2(dx)}\Big).$$

We note that as $\partial_\theta u_0 = 0$, where in an abuse of notation u_0 here stands for the zero mode, the estimate when $k = 0$ follows trivially from (2.3). Thus, it remains to show

$$\int_0^T \|\chi(x) k u_k\|^2_{L^2(\mathbb{R})}\,dt \leqslant C_T\Big(\| \langle k\rangle^{\beta(m_1,m_2)} u_{0,k}\|^2_{L^2} + \|u_{0,k}\|^2_{H^{1/2}}\Big), \quad |k| \geqslant 1$$

for some $\chi \in \mathcal{C}^\infty_c(\mathbb{R})$ with $\chi(x) \equiv 1$ in a neighborhood of $x = 0$ and also in a neighborhood of $x = 1$.

In the sequel, we shall be working with a fixed k and as such shall drop the subscript notation. Set

$$P_k = D_x^2 + A^{-2}(x)k^2 + V_1(x).$$

Notice that P_k is merely $-\widetilde{\Delta}$ applied to the kth mode. We fix an even function $\psi \in \mathcal{C}^\infty_c(\mathbb{R})$ which is 1 for $|r| \le \epsilon$ and vanishes for $|r| \ge 2\epsilon$ where $\epsilon > 0$ will be determined later. Then let

$$u = u_{hi} + u_{lo}, \quad u_{hi} = \psi(D_x/k)u.$$

2.1.3. *Low frequency estimate.* We first examine u_{lo} and reduce estimating it to understanding a bound for u_{hi}. We observe that $u_{lo} = (1 - \psi(D_x/k))u$ solves

$$(D_t + P_k)u_{lo} = -[P_k, \psi(D_x/k)]u = k\langle x\rangle^{-1} L_k \langle x\rangle^{-2} \tilde{\psi}(D_x/k)u$$

where L_k is L^2 bounded uniformly in k and $\tilde{\psi} \in \mathcal{C}^\infty_c$ which is identity on the support of ψ.

Choosing the same multiplier B, replacing $-\widetilde{\Delta}$ with the self-adjoint P_k, and integrating (2.2) yields

$$\Big|\int_0^T \langle [P_k, B]u_{lo}, u_{lo}\rangle\,dt\Big| \leqslant C\Big(|\langle u_{lo}, \arctan(x)\partial_x u_{lo}\rangle|\Big|_0^T + |\langle u_{lo}, \langle x\rangle^{-1} u_{lo}\rangle|\Big|_0^T + \Big|\int_0^T \langle \langle x\rangle^{-1} k L_k \langle x\rangle^{-2}\tilde{\psi}(D_x/k)u, B u_{lo}\rangle\,dt\Big|\Big). \tag{2.4}$$

Continuing to argue as above shows that

$$\int_0^T \|\langle x\rangle^{-1}\partial_x u_{lo}\|^2_{L^2}\,dt \leqslant C_T\Big(\|u_0\|^2_{H^{1/2}} + \Big|\int_0^T \langle \langle x\rangle^{-1} k L_k \langle x\rangle^{-2}\tilde{\psi}(D_x/k)u, B u_{lo}\rangle\,dt\Big|\Big).$$

Applying the Schwarz inequality to the last term and bootstrapping, we obtain

$$\int_0^T \|\langle x\rangle^{-1}\partial_x u_{lo}\|^2_{L^2}\,dt \leqslant C_T\Big(\|u_0\|^2_{H^{1/2}} + \int_0^T \|k\langle x\rangle^{-2}\tilde{\psi}(D_x/k)u\|^2_{L^2}\,dt\Big). \tag{2.5}$$

The frequency cutoff guarantees that

$$\int_0^T \|\langle x\rangle^{-1} k u_{lo}\|^2_{L^2}\,dt \leqslant C \int_0^T \|\langle x\rangle^{-1}\partial_x u_{lo}\|^2_{L^2}\,dt.$$

As (2.3) provides control on the last term in (2.5) away from $x = 0$ and $x = 1$, it suffices to prove

$$\int_0^T \|\chi k\tilde{\psi}(D_x/k)u\|_{L^2}^2\,dt \leqslant C_T\|k^{\beta(m_1,m_2)}u_0\|_{L^2}^2.$$

Here χ is a cutoff which is 1 in a neighborhood of the trapped geodesics at $x = 0$ and $x = 1$. The desired bound, thus, will follow once u_{hi} is controlled as the precise choice of cutoff ψ is inessential.

2.1.4. *High frequency estimate.* It remains to estimate u_{hi} in the vicinity of $x = 0$ and $x = 1$. We fix a cutoff $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ which is 1 in a neighborhood of $x = 0$ and in a neighborhood of $x = 1$. Let

$$F(t)g = \chi(x)\psi(D_k/k)k^r e^{-itP_k}g,$$

where the constant $r > 0$ will be determined later. We seek to determine r so that $F : L_x^2 \to L^2([0,T]; L_x^2)$. The resulting inequality,

$$\|k^{1-r}F(t)u_0\|_{L^2([0,T];L_x^2)} \leqslant C_T\|k^{1-r}u_0\|_{L^2}, \tag{2.6}$$

is a local smoothing estimate. F is such a mapping if and only if $FF^* : L^2L^2 \to L^2L^2$, where we have abbreviated $L^2([0,T]; L_x^2) = L^2L^2$. A straightforward computation shows that

$$FF^*f(t,x) = \chi(x)\psi(D_x/k)k^{2r}\int_0^T e^{-i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)\,ds,$$

and

$$\|FF^*f\|_{L^2L^2} \leqslant C_T\|f\|_{L^2L^2}$$

is the desired estimate. We write $FF^*f(t,x) = \chi(x)\psi(D_x/k)(v_1 + v_2)$ where

$$v_1 = k^{2r}\int_0^t e^{-i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)\,ds,$$
$$v_2 = k^{2r}\int_t^T e^{-i(t-s)P_k}\psi(D_x/k)\chi(x)f(s,x)\,ds.$$

Thus,

$$(D_t + P_k)v_l = (-1)^l ik^{2r}\psi(D_x/k)\chi(x)f, \quad l = 1,2$$

and

$$\|\chi\psi v_l\|_{L^2L^2} \leqslant C_T\|f\|_{L^2L^2}$$

would imply the desired estimate. By Plancherel's theorem, this is equivalent to showing

$$\|\chi\psi\hat{v}_l\|_{L^2L^2} \leqslant C_T\|\hat{f}\|_{L^2L^2}$$

where $\hat{f}$ denotes the Fourier transform of f in the time variable. I.e., we are required to show that

$$\|\chi\psi k^{2r}(\tau \pm i0 + P_k)^{-1}\psi\chi\|_{L_x^2\to L_x^2} = O(1)$$

uniformly in τ. Setting, as above, $-z = \tau k^{-2}$, $h = k^{-1}$, and $V = A^{-2}(x) + h^2V_1(x)$, we need

$$\|\chi(x)\psi(hD_x)(-z \pm i0 + (hD_x)^2 + V)^{-1}\psi(hD_x)\chi(x)\|_{L^2\to L^2} \leqslant Ch^{-2(1-r)}. \tag{2.7}$$

We recall

$$P = (hD_x)^2 + V$$

and shall use gluing techniques to reduce proving

$$\|\rho_{-s}(P-z)u\|_{L^2} \geqslant ch^{-2\beta(m_1,m_2)}\|\rho_s u\|_{L^2}, \quad s<-1/2 \tag{2.8}$$

which implies (2.7) with $1-r=\beta(m_1,m_2)$ as desired, to proving microlocal invertibility estimates near the trapped sets. In (2.8), ρ_s is a smooth function that is identity on a large compact set and is equivalent to $\langle x\rangle^s$ near infinity.

The gluing techniques that we shall employ are outlined in [Chr13]. See also [Chr08, Proposition 2.2] and [DV12].

Recall that we are working in $T^*\mathbb{R}$ with principal symbol $p=\xi^2+V(x)$ where the potential $V(x)$ is a short range perturbation of x^{-2} and has critical points at precisely $x=0,1$. The critical point at $x=0$ is a maximum with value 1 while the critical point at 1 is an inflection point with potential value C_1^{-1}. This means that in terms of the Hamiltonian vector field, H_p, the level set $\{p=1\}$ contains the critical point $(0,0)$ and the level set $\{p=C_1^{-1}\}$ contains the critical point $(1,0)$. Furthermore, $\pm V'(x)\le 0$ for $\pm x\ge 0$ with equality only at these critical points.

As in [Chr13], we fix a few cutoffs. Let $M>1$ be sufficiently large so that there is a symbol p_0 such that $p_0=p$ for $|x|\ge M-1$ and the operator P_0 associated to symbol p_0 satisfies

$$\|\rho_{-s}(P_0-z)u\|_{L^2} \geqslant c\frac{h}{\log(1/h)}\|\rho_s u\|_{L^2}.$$

Such a P_0 is, e.g., the $m=1$ case of [CW11] and such bounds follow from [Chr07, Chr11]. Here ρ_s is a smooth function such that $\rho_s>0$, $\rho_s(x)\equiv 1$ on a neighborhood of $\{|x|\le 2M\}$, and $\rho_s\equiv\langle x\rangle^s$ for x sufficiently large. We choose $\Gamma\in\mathcal{C}_c^\infty(\mathbb{R})$ with $\Gamma\equiv 1$ on $\{|x|\le M-1\}$ with support in $\{|x|\le M\}$. In particular, $p=p_0$ on $\operatorname{supp}(1-\Gamma)$.

Let

$$\Lambda(r) := \int_0^r \langle t\rangle^{-1-\epsilon_0}\,dt, \tag{2.9}$$

for some fixed $\epsilon_0>0$, which is a function chosen to be globally bounded with positive derivative and $\Lambda(r)\sim r$ near $r=0$. Then let

$$a(x,\xi)=\Lambda(x)\Lambda(\xi),$$

so that

$$\begin{aligned} H_p a &= (2\xi\partial_x - V'(x)\partial_\xi)a \\ &= 2\xi\Lambda(\xi)\Lambda'(x) - V'(x)\Lambda(x)\Lambda'(\xi). \end{aligned}$$

Since $\pm V'(x)<0$ for $\pm x>0$, $x\neq 1$, for any $\epsilon>0$ we have

$$H_p a \ge c_0 > 0, \quad |x|\in[\epsilon/2, 1-(\epsilon/2)]\cup[1+(\epsilon/2), M]. \tag{2.10}$$

We further have

$$H_p a \ge c_0' > 0, \quad |\xi|\ge\delta>0 \text{ and } |x|\le M. \tag{2.11}$$

For $j=0,1$, let $\Gamma_j(x)$ be equal to 1 for $|x-j|\leqslant\epsilon/2$ with support in $\{|x-j|\leqslant\epsilon\}$. And set

$$\Gamma_2=\Gamma-\Gamma_0-\Gamma_1$$

so that Γ_2 is supported where $|x|\in[\epsilon/2,1-(\epsilon/2)]\cup[1+(\epsilon/2),M]$.

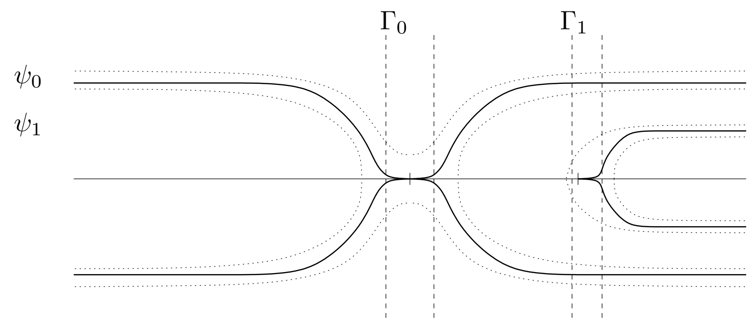

FIGURE 4. The cutoff functions used to apply [Chr13, Appendix].

We may use a commutator argument to prove the necessary microlocal black box estimate for Γ_2. Indeed, for any $z \in \mathbb{R}$, using (2.10),

$$\begin{aligned} 2\operatorname{Im}\langle (P-z)\Gamma_2 u, a^w \Gamma_2 u\rangle &= -i\langle (P-z)\Gamma_2 u, a^w\Gamma_2 u\rangle + i\langle a^w \Gamma_2 u, (P-z)\Gamma_2 u\rangle \\ &= i\langle [P, a^w]\Gamma_2 u, \Gamma_2 u\rangle \geqslant c_1 h \langle \Gamma_2 u, \Gamma_2 u\rangle \end{aligned}$$

for some $c_1 > 0$. Then

$$c_1 h \|\Gamma_2 u\|^2 \leq 2\|(P-z)\Gamma_2 u\| \|a^w \Gamma_2 u\| \leq C\|(P-z)\Gamma_2 u\| \|\Gamma_2 u\|,$$

so that

$$\|\Gamma_2 u\| \leq C' h^{-1} \|(P-z)\Gamma_2 u\|,$$

as desired.

We now choose two microlocal cutoffs. For $j = 0, 1$, let $\psi_j = \psi_j(\xi^2 + V(x))$ be functions of the principal symbol p. For some $\delta > 0$ to be fixed momentarily, assume $\psi_0 \equiv 1$ for $\{|p-1| \leqslant \delta\}$ with slightly larger support and similarly $\psi_1 \equiv 1$ for $\{|p - C_1^{-1}| \leqslant \delta\}$ with slightly larger support. The parameter $\delta > 0$ may now be fixed, depending on $\varepsilon > 0$, so that $\psi_1 \equiv 1$ on $\operatorname{supp}(\Gamma_1) \cap \{\xi \equiv 0\}$. These cutoffs are depicted in Figure 4. Repeating the commutator argument above but instead using (2.11) allows us to conclude

$$\|\Gamma_0(1-\psi_0)u\| \leq Ch^{-1}\|(P-z)\Gamma_0(1-\psi_0)u\|$$

and

$$\|\Gamma_1(1-\psi_1)u\| \leq Ch^{-1}\|(P-z)\Gamma_1(1-\psi_1)u\|.$$

We also include a separate figure (Figure 5) that illustrates that such a microlocalization can be carried out in the case described in Remark 1.3.

We may now conclude (2.8) provided that we can establish such microlocal invertibility estimates for $\Gamma_j \psi_j u$.

The invertibility estimate near $(0,0)$ has been proved in [Chr07, Chr11] for $m_1 = 1$ and in [CW11] for $m_1 > 1$. For convenience, this is restated

Lemma 2.2. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x,\xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|(P-z)\varphi^w u\| \geqslant C_\epsilon h^{2m_1/(m_1+1)} \|\varphi^w u\|, \;\; z \in [1-\epsilon, 1+\epsilon], \tag{2.12}$$

if $m_1 > 1$. If $m_1 = 1$, then $h^{2m_1/(m_1+1)}$ is replaced by $h/(\log(1/h))$.

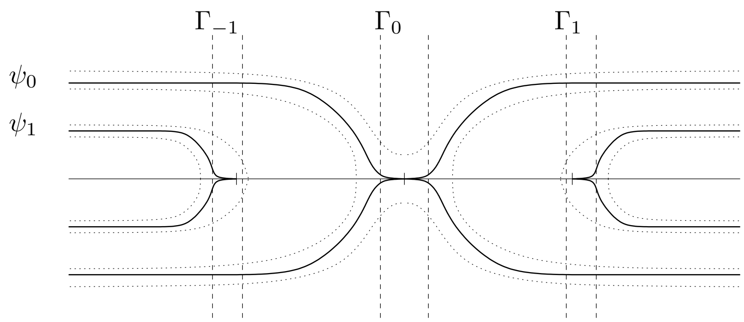

FIGURE 5. The cutoff functions in the case described in Remark 1.3.

We need only prove the corresponding estimate near $(1, 0)$. This is also used to prove Theorem 2.

Lemma 2.3. *For $\epsilon > 0$ sufficiently small, let $\varphi \in \mathcal{S}(T^*\mathbb{R})$ have compact support in $\{|(x-1, \xi)| \leqslant \epsilon\}$. Then there exists $C_\epsilon > 0$ such that*

$$\|(P-z)\varphi^w u\| \geqslant C_\epsilon h^{(4m_2+2)/(2m_2+3)} \|\varphi^w u\|, \quad z \in [C_1^{-1} - \epsilon, C_1^{-1} + \epsilon]. \tag{2.13}$$

The proof of this estimate proceeds through several steps. First, we rescale the principal symbol of P to introduce a calculus of two parameters. We then quantize in the second parameter which eventually will be fixed as a constant in the problem. This technique has been used in [SZ02, SZ07], [Chr07, Chr11], [CW11].

2.2. **The two parameter calculus.** Before we proceed to the proof of Lemma 2.3, we shall first review some facts about the two parameter calculus. These ideas were introduced in [SZ07], and we shall employ the generalizations proved in [CW11].

We set

$$\mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}(T^*(\mathbb{R}^n)) := $$
$$= \left\{ a \in \mathcal{C}^\infty\left(\mathbb{R}^n \times (\mathbb{R}^n)^* \times (0,1]^2\right) : \right.$$
$$\left. \left|\partial_x^\rho \partial_\xi^\gamma a(x, \xi; h, \tilde{h})\right| \leqslant C_{\rho\gamma} h^{-m} \tilde{h}^{-\tilde{m}} \left(\frac{\tilde{h}}{h}\right)^{\alpha|\rho| + \beta|\gamma|} \langle \xi \rangle^{k-|\gamma|} \right\},$$

when $\alpha \in [0, 1]$ and $\beta \leq 1 - \alpha$. Throughout, we take $\tilde{h} \geq h$. We abbreviate $\mathcal{S}^{0,0,0}_{\alpha,\beta}$ by $\mathcal{S}_{\alpha,\beta}$. The focus shall be on the marginal case $\alpha + \beta = 1$. In particular, even in this marginal case, we have that $a \in \mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}$ and $b \in \mathcal{S}^{k',m',\tilde{m}'}_{\alpha,\beta}$ implies that

$$\operatorname{Op}_h^w(a) \circ \operatorname{Op}_h^w(b) = \operatorname{Op}_h^w(c)$$

for some symbol $c \in \mathcal{S}^{k+k', m+m', \tilde{m}+\tilde{m}'}_{\alpha,\beta}$.

We also have the following expansion. This is from [SZ07, Lemma 3.6] in the case that $\alpha = \beta = 1/2$ and from [CW11] in the more general case.

Lemma 2.4. *Suppose that $a, b \in \mathcal{S}_{\alpha,\beta}$, and that $c^w = a^w \circ b^w$. Then*

$$c(x,\xi) = \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k a(x,\xi)b(y,\eta)|_{x=y,\xi=\eta} + e_N(x,\xi)\,, \tag{2.14}$$

where for some M

$$\begin{aligned} |\partial^\gamma e_N| &\leqslant C_N h^{N+1} \\ &\times \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n \\ (y,\eta)\in T^*\mathbb{R}^n}} \sup_{|\rho|\leqslant M\,,\rho\in\mathbb{N}^{4n}} \left|\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{N+1}a(x,\xi)b(y,\eta)\right|\,, \end{aligned} \tag{2.15}$$

where $\sigma(D) = \sigma(D_x, D_\xi; D_y, D_\eta)$ as usual, and

$$\Gamma_{\alpha,\beta,\rho,\gamma}(D) = (h^\alpha \partial_{(x,y)}, h^\beta \partial_{(\xi,\eta)}))^\rho \partial^{\gamma_1}_{(x,\xi)} \partial^{\gamma_2}_{(y,\eta)}.$$

With the scaling of coordinates

$$(x,\xi) = \mathcal{B}(X,\Xi) = ((h/\tilde{h})^\alpha X, (h/\tilde{h})^\beta \Xi), \tag{2.16}$$

it follows that if $a \in \mathcal{S}^{k,m,\tilde{m}}_{\alpha,\beta}$ then $a \circ \mathcal{B} \in \mathcal{S}^{k,m,\tilde{m}}_{0,0}$. Moreover, the unitary operator $T_{h,\tilde{h}}u(X) = \left(h/\tilde{h}\right)^{\frac{n\alpha}{2}} u\left(\left(h/\tilde{h}\right)^\alpha X\right)$, relates the quantizations

$$\operatorname{Op}^w_{\tilde{h}}(a \circ \mathcal{B}) T_{h,\tilde{h}} u = T_{h,\tilde{h}} \operatorname{Op}^w_h(a) u. \tag{2.17}$$

2.3. **Proof of Lemma 2.3.** Due to the cutoff φ^w, we are working microlocally in $\{|(x-1,\xi)| \leqslant \epsilon\}$. We notice that it suffices to demonstrate (2.13) for $P-z$ replaced by $Q_1 = P - h^2 V_1 - z$ as V_1 is bounded in this region and $(4m_2+2)/(2m_2+3) < 2$.

Let

$$q_1 = \xi^2 + A^{-2} - z$$

be the principal symbol of Q_1. Applying Taylor's theorem about $x = 1$ to A^{-2}, we have

$$q_1 = \xi^2 - \frac{c_2}{C_1^2}(x-1)^{2m_2+1}(1 + \tilde{a}(x)) - z_1$$

where $z_1 = z - C_1^{-1} \in [-\epsilon, \epsilon]$ and $\tilde{a}(x) = \mathcal{O}(|x-1|^1)$. For our specific example, this error is $\mathcal{O}(|x-1|^{2m_2+1})$, but more generally, when one merely assumes the order of vanishing at the critical point, the error is as we have listed. The Hamilton vector field H associated to the symbol q_1 is

$$\mathsf{H} = 2\xi\partial_x + \left((2m_2+1)\frac{c_2}{C_1^2}(x-1)^{2m_2} + \mathcal{O}(|x-1|^{2m_2+1})\right)\partial_\xi.$$

We introduce the new variables

$$X - 1 = \frac{x-1}{(h/\tilde{h})^\alpha}, \quad \Xi = \frac{\xi}{(h/\tilde{h})^\beta},$$

where

$$\alpha = \frac{2}{2m_2+3}, \quad \beta = \frac{2m_2+1}{2m_2+3},$$

and, as above, we shall use $\mathcal{B}$ to denote the map $\mathcal{B}(X-1,\Xi) = (x-1,\xi)$. In these new coordinates, we record that

$$\begin{aligned} \mathsf{H} = (h/\tilde{h})^{\frac{2m_2-1}{2m_2+3}}\Big(2\Xi\partial_X &+ (2m_2+1)\frac{c_2}{C_1^2}(X-1)^{2m_2}\partial_\Xi \\ &+ \mathcal{O}((h/\tilde{h})^\alpha |X-1|^{2m_2+1})\partial_\Xi\Big). \end{aligned} \tag{2.18}$$

We recall the definition (2.9) of $\Lambda(r)$, and we similarly set

$$\Lambda_2(r) = 1 + \int_{-\infty}^{r} \langle t\rangle^{-1-\epsilon_0}\, dt.$$

Then, for a cutoff function $\chi(s)$ which is identity for $|s| < \delta_1$ and vanishes for $|s| > 2\delta_1$, we introduce

$$a(x,\xi;h,\tilde{h}) = \Lambda(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi),$$

where $\delta_1 > 0$ is another parameter which will be fixed shortly. As $\tilde{h} \geq h$, we have that

$$\left|\partial_X^{\ell_1}\partial_\Xi^{\ell_2} a\right| \leqslant C_{\ell_1,\ell_2}$$

for any $\ell_1, \ell_2 \geq 0$. We compute

$$\mathsf{H}(a) = (h/\tilde{h})^{\frac{2m_2-1}{2m_2+3}} g(x,\xi;h,\tilde{h}) + r(x,\xi;h,\tilde{h})$$

where

$$g = \chi(x-1)\chi(\xi)\Big(2\Lambda(\Xi)\Xi\langle X-1\rangle^{-1-\epsilon_0} + (2m_2+1)\frac{c_2}{C_1^2}(X-1)^{2m_2}\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)(1+\mathcal{O}(|x-1|^1))\Big) \tag{2.19}$$

and

$$\operatorname{supp} r \subset \{|x-1| > \delta_1\} \cup \{|\xi| > \delta_1\}.$$

We first seek to show that the following lemma from [CW11] may be applied to g:

Lemma 2.5. *Let a real-valued symbol $\tilde{g}(x,\xi;h)$ satisfy*

$$\tilde{g}(x,\xi;h) = \begin{cases} c(\xi^2 + x^{2m})(1+r_2), & \xi^2 + x^2 \leqslant 1 \\ b(x,\xi;h), & \xi^2 + x^2 \geqslant 1, \end{cases}$$

where $c > 0$ is constant, $r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic. Then there exists $c_0 > 0$ such that

$$\langle \operatorname{Op}_h^w(\tilde{g})u, u\rangle \geqslant c_0 h^{2m/(m+1)} \|u\|_{L^2}^2$$

for h sufficiently small.

In the sequel, we shall only be applying the above to functions which are microlocally cutoff to the set where $\chi(x-1)\chi(\xi) \equiv 1$. As the errors off this set will be $\mathcal{O}(h^\infty)$, we shall assume that $|x-1| \leq \delta_1$ and $|\xi| \leq \delta_1$ throughout this discussion.

Over $|(X-1,\Xi)| \leqslant 1$, we have $\Lambda(\Xi) \sim \Xi$, $\Lambda_2(X-1) \sim 1$, and $\langle X-1\rangle^{-1-\varepsilon_0} \sim 1$. Thus, the term g, given in (2.19), of $\mathsf{H}(a)$ is bounded below by a multiple of $\Xi^2 + (X-1)^{2m_2}$.

We next consider $|(X-1,\Xi)| \geqslant 1$. Since $\operatorname{sgn}\Lambda(s) = \operatorname{sgn}(s)$, when $|\Xi| \geqslant \max(|X-1|^{1+\epsilon_0}, 1/4)$, then

$$g \geqslant 2\Lambda(\Xi)\langle X-1\rangle^{-1-\epsilon_0}\Xi \gtrsim \frac{|\Xi|}{\langle\Xi\rangle} \geqslant C > 0.$$

For $|X-1|^{1+\epsilon_0} \geqslant \max(|\Xi|, 1/4)$, we have

$$g \geqslant C'\langle\Xi\rangle^{-1-\epsilon_0}\Lambda_2(X-1)(X-1)^{2m_2} \gtrsim |X-1|^{-(1+\epsilon_0)^2}|X-1|^{2m_2} \geqslant C'' > 0,$$

provided $(1+\epsilon_0)^2 < 2m_2$. In the region of interest $|(X-1,\Xi)| \geqslant 1$, the larger of $|\Xi|$ and $|X-1|^{1+\epsilon_0}$ is assuredly greater than $1/4$ if $\epsilon_0 > 0$ is sufficiently small. Hence, we have shown that

$$g \geqslant C > 0 \quad \text{in } \{\Xi^2 + (X-1)^2 \geqslant 1\}.$$

Recapping, we have found that

$$\mathsf{H}(a) = (h/\tilde{h})^{\frac{2m_2-1}{2m_2+3}} g + r$$

with

$$r = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}((h/\tilde{h})^{(2m_2-1)/(2m_2+3)}((h/\tilde{h})^{\alpha}|\Xi| + (h/\tilde{h})^{\beta}|X-1|^{2m_2}))$$

supported as above and

$$g(X,\Xi;h) = \begin{cases} c(\Xi^2 + (X-1)^{2m_2})(1+r_2), & \Xi^2 + (X-1)^2 \leqslant 1 \\ b, & \Xi^2 + (X-1)^2 \geqslant 1, \end{cases}$$

where $c > 0$ is a constant, $r_2 = \mathcal{O}_{\mathcal{S}_{\alpha,\beta}}(\delta_1)$, and $b > 0$ is elliptic.

By translating, using the blowdown map $\mathcal{B}$, and relating the quantizations as in the previous section, we may use Lemma 2.5 to obtain a similar bound on g.

Lemma 2.6. *For g given by* (2.19) *and $\tilde{h} > 0$ sufficiently small, there exists $c > 0$ such that*

$$\|\mathrm{Op}_h^w(g \circ \mathcal{B}^{-1})\|_{L^2 \to L^2} > c\tilde{h}^{2m_2/(m_2+1)},$$

uniformly as $h \downarrow 0$.

The proof of this lemma follows exactly as that in [CW11] and is, thus, omitted.

Before completing the proof of Lemma 2.3, we need the following lemma about the lower order terms in the expansion of the commutator of Q_1 and a^w.

Lemma 2.7. *The symbol expansion of $[Q_1, a^w]$ in the h-Weyl calculus is of the form*

$$[Q_1, a^w] = \mathrm{Op}_h^w\Bigg(\Big(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\Big)(q_1(x,\xi)a(y,\eta) - q_1(y,\eta)a(x,\xi))\Big|_{x=y,\xi=\eta} + e(x,\xi) + r_3(x,\xi)\Bigg),$$

where r_3 is supported in $\{|(x,\xi) \geqslant \delta_1\}$ and e satisfies

$$\|\mathrm{Op}_h^w(e)\|_{L^2\to L^2} \leqslant C\tilde{h}^{\frac{2m_2+7}{2m_2+3} - \frac{2m_2}{m_2+1}} h^{\frac{4m_2+2}{2m_2+3}}\Big(\|\mathrm{Op}_h^w(g\circ\mathcal{B}^{-1})\|_{L^2\to L^2} + \mathcal{O}(\tilde{h}^{2+\frac{2m_2}{m_2+1}})\Big),$$

with g given by (2.19).

Proof. Since everything is in the Weyl calculus, only the odd terms in the exponential composition expansion are non-zero. In accordance with Lemma 2.4, we set

$$\begin{aligned} e(x,\xi) &\\ &= \chi(x-1)\chi(\xi) \\ &\quad \times \sum_{k=1}^{m_2-1} \frac{2}{(2k+1)!}\Big(\frac{ih}{2}\sigma(D)\Big)^{2k+1} q_1(x,\xi)\Lambda((\tilde{h}/h)^\beta\eta)\Lambda_2((\tilde{h}/h)^\alpha(y-1))\Big|_{\substack{x=y\\ \xi=\eta}} \\ &\quad + \chi(\xi)\chi(x-1)e_{2m_2}(x,\xi). \end{aligned}$$

Here we have extracted the terms in the expansion where derivatives fall on the cutoff $\chi(\eta)$ of a as these terms have supports compatible with r_3. For convenience, however, e_{2m_2} denotes the full error in the expansion of $[Q_1, a^w]$.

Recalling that $q_1(x,\xi) = \xi^2 - (x-1)^{2m_2+1}(1+\tilde{a}(x))$, it follows that

$$\begin{aligned} \tilde{e}_k &:= h^{2k+1}\chi(x-1)\chi(\xi)\sigma(D)^{2k+1}q_1(x,\xi)\Lambda((\tilde{h}/h)^\beta\eta)\Lambda_2((\tilde{h}/h)^\alpha(y-1))\Big|_{\substack{x=y\\ \xi=\eta}} \\ &= h^{2k+1}\chi(x-1)\chi(\xi)D_x^{2k+1}q_1(x,\xi)D_\eta^{2k+1}\Lambda((\tilde{h}/h)^\beta\eta)\Lambda_2((\tilde{h}/h)^\alpha(y-1))\Big|_{\substack{x=y\\ \xi=\eta}} \\ &= ch^{2k+1}(x-1)^{2m_2+1-(2k+1)}(1+\mathcal{O}((x-1)^{2m_2+1})) \\ &\qquad \times (\tilde{h}/h)^{(2k+1)\beta}\Lambda^{(2k+1)}((\tilde{h}/h)^\beta\xi) \\ &\qquad \times \Lambda_2((\tilde{h}/h)^\alpha(x-1))\chi(x-1)\chi(\xi) \end{aligned}$$

for $1 \le k \le m_2 - 1$.

In order to estimate e, we first estimate each $\tilde{e}_k$, $1 \le k \le m_2 - 1$, using conjugation to the 2-parameter calculus. We have

$$\|\mathrm{Op}_h^w(\tilde{e}_k)u\|_{L^2} = \|T_{h,\tilde{h}}\mathrm{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1}T_{h,\tilde{h}}u\|_{L^2} \leqslant \|T_{h,\tilde{h}}\mathrm{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}\|_{L^2\to L^2}\|u\|_{L^2}$$

since $T_{h,\tilde{h}}$ is unitary. We recall that $T_{h,\tilde{h}}\mathrm{Op}_h^w(\tilde{e}_k)T_{h,\tilde{h}}^{-1} = \mathrm{Op}_{\tilde{h}}^w(\tilde{e}_k \circ \mathcal{B})$ and note that

$$\begin{aligned} \tilde{e}_k \circ \mathcal{B} = ch^{2k+1}(h/\tilde{h})^{(2m_2+1-(2k+1))\alpha-(2k+1)\beta}(X-1)^{2m_2+1-(2k+1)} \\ \times (1+\mathcal{O}((x-1)^1))\Lambda^{(2k+1)}(\Xi)\Lambda_2(X-1)\chi(x-1)\chi(\xi), \end{aligned}$$

which can be estimated by

$$Ch^{\frac{4m_2+2}{2m_2+3}}\tilde{h}^{\frac{2m_2+7}{2m_2+3}}\tilde{h}^{2(k-1)}(X-1)^{(2m_2+1)-(2k+1)}\Lambda^{(2k+1)}(\Xi)\chi(x-1)\chi(\xi).$$

On $\{|X-1| \leqslant 1\}$, we have that

$$(X-1)^{(2m_2+1)-(2k+1)}\Lambda^{(2k+1)}(\Xi)\chi(x-1)\chi(\xi)$$

is bounded, and thus,

$$\|\mathrm{Op}_{\tilde{h}}^w(\tilde{e}_k)\|_{L^2\to L^2} \le C\tilde{h}^{-2m_2/(m_2+1)}\|\mathrm{Op}_h^w(g\circ\mathcal{B}^{-1})\|_{L^2\to L^2}$$

by Lemma 2.6. While on $\{|X-1| \geqslant 1\}$, we have $\tilde{e}_k \le g$, and thus

$$\|\mathrm{Op}_{\tilde{h}}^w(\tilde{e}_k)\|_{L^2\to L^2} \le \|\mathrm{Op}_{\tilde{h}}^w(g)\|_{L^2\to L^2} + O(\tilde{h}^2) \le \|\mathrm{Op}_h^w(g\circ\mathcal{B}^{-1})\|_{L^2\to L^2} + O(\tilde{h}^2).$$

For e_{2m_2}, by the standard L^2 continuity theorem of h-pseudodifferential operators, it suffices to estimate a finite number of derivatives of the error e_{2m_2}. We note the bound of Lemma 2.4

$$|\partial^\gamma e_{2m_2}| \leqslant Ch^{2m_2+1} \sum_{\gamma_1+\gamma_2=\gamma} \sup_{\substack{(x,\xi)\in T^*\mathbb{R}^n \\ (y,\eta)\in T^*\mathbb{R}^n \\ \rho\in\mathbb{N}^{4n}, |\rho|\leq M}} |\Gamma_{\alpha,\beta,\rho,\gamma}(D)(\sigma(D))^{2m_2+1} q_1(x,\xi)a(y,\eta)|.$$

We have

$$\begin{aligned}(\sigma(D))^{2m_2+1} & q_1(x,\xi)a(y,\eta) \\ &= c(1+\mathcal{O}(x-1)^1)\chi(y-1)\Lambda((\tilde{h}/h)^\alpha(y-1))D_\eta^{2m_2+1}[\Lambda((\tilde{h}/h)^\beta\eta)\chi(\eta)].\end{aligned}$$

The last factor is $\mathcal{O}((\tilde{h}/h)^{(2m_2+1)\beta})$. Moreover, the derivatives $h^\beta\partial_\eta$ and $h^\alpha\partial_y$ preserve the order of h and increase the order of $\tilde{h}$, while the other derivatives lead to higher powers of $h/\tilde{h} \ll 1$. It, thus, follows that $|\partial^\gamma(\chi(x-1)\chi(\xi)e_{2m_2})|$ is

$$\mathcal{O}(h^{(4m_2+2)/(2m_2+3)}\tilde{h}^{(2m_2+1)^2/(2m_2+3)}),$$

and thus, when also combined with Lemma 2.4 satisfies the given bound. □

We now complete the proof of Lemma 2.3. We set $v = \varphi^w u$ where φ has support where $\chi(x)\chi(\xi) = 1$, and in particular, away from the support of r_3.

Then Lemmas 2.6 and 2.7 yield

$$\begin{aligned} i\langle[Q_1, a^w]v, v\rangle &= h\langle \mathrm{Op}_h^w(\mathsf{H}(a))v, v\rangle + \langle \mathrm{Op}_h^w(e)u, u\rangle + \mathcal{O}(h^\infty)\|v\|_{L^2}^2 \\ &= h(h/\tilde{h})^{\frac{2m_2-1}{2m_2+3}}\big\langle \mathrm{Op}_h^w(g\circ\mathcal{B}^{-1})v, v\big\rangle + \langle \mathrm{Op}_h^w(e)u, u\rangle + \mathcal{O}(h^\infty)\|v\|_{L^2}^2 \\ &= h^{\frac{4m_2+2}{2m_2+3}}\big(\tilde{h}^{-\frac{2m_2-1}{2m_2+3}} + \mathcal{O}(\tilde{h}^{\frac{2m_2+7}{2m_2+3}-\frac{2m_2}{m_2+1}})\big)\big\langle \mathrm{Op}_h^w(g\circ\mathcal{B}^{-1})v, v\big\rangle \\ &\qquad + (\mathcal{O}(h^\infty) + \mathcal{O}(\tilde{h}^{2+\frac{2m_2}{m_2+1}}))\|v\|_{L^2}^2 \\ &\geqslant Ch^{\frac{4m_2+2}{2m_2+3}}\tilde{h}^{1+\frac{4}{2m_2+3}-\frac{2}{m_2+1}}\|v\|_{L^2}^2, \end{aligned}$$

for $\tilde{h}$ sufficiently small. The Schwarz inequality and the L^2 continuity theorem for h-pseudodifferential operators guarantees

$$\big|\langle[Q_1, a^w]v, v\rangle\big| \leqslant C\|Q_1 v\|_{L^2}\|v\|_{L^2},$$

and thus the desired bound with $1 \gg \tilde{h} > 0$ fixed. □

3. Quasimodes

We end by constructing quasimodes near $(1,0)$ in phase space and use these to saturate the estimate of Proposition 2.1, and hence that of Theorem 1. The proofs follow from straightforward modifications of those in [CW11]. We, thus, only provide a terse description.

Quasimodes were already constructed near $(0,0)$ in [CW11]. We focus only on the construction near the inflection point. We let

$$\widetilde{P} = -h^2\partial_x^2 - c_2(x-1)^{2m_2+1}$$

near $x = 1$ and construct quasimodes that are localized to a small neighborhood of $x = 1$.

We set

$$\gamma = \frac{4m+2}{2m+3},$$

$E = (\alpha + i\beta)h^\gamma$ where $\alpha, \beta > 0$ and are independent of h, and

$$\varpi(x) = \int_1^x (E + c_2(y-1)^{2m_2+1})^{1/2} dy,$$

where the branch of the square root is chosen to have positive imaginary part. Letting

$$u(x) = (\varpi')^{-1/2} e^{i\varpi/h},$$

we see that

$$(hD)^2 u = (\varpi')^2 u + fu,$$

where

$$f = -h^2 \left(\frac{3}{4} (\varpi')^{-2} (\varpi'')^2 - \frac{1}{2} (\varpi')^{-1} \varpi''' \right).$$

Straightforward modifications of the proof contained in [CW11, Lemma 3.1] yield the following:

Lemma 3.1. *The phase function ϖ satisfies the following properties:*

(i): *There exists $C > 0$ independent of h such that*

$$|\operatorname{Im} \varpi| \leqslant Ch.$$

In particular, if $|x-1| \leqslant Ch^{\gamma/(2m_2+1)}$, $|\operatorname{Im}\varpi| \leqslant C'$ for some $C' > 0$ independent of h.

(ii): *There exists $C > 0$ independent of h such that if $\delta > 0$ is sufficiently small and $|x-1| \leqslant \delta h^{\gamma/(2m_2+1)}$, then*

$$C^{-1} h^{\gamma/2} \leqslant |\varpi'(x)| \leqslant Ch^{\gamma/2}.$$

(iii):

$$\begin{aligned} \varpi' &= (E + c_2(x-1)^{2m_2+1})^{1/2}, \\ \varpi'' &= \frac{1}{2} c_2 (2m_2+1)(x-1)^{2m_2} (\varpi')^{-1}, \\ \varpi''' &= \Big(\frac{1}{2} c_2 (2m_2+1)(2m_2)(E(x-1)^{2m_2-1} + c_2(x-1)^{4m_2}) \\ &\qquad - \frac{1}{4} c_2^2 (2m_2+1)^2 (x-1)^{4m_2} \Big) (\varpi')^{-3}. \end{aligned}$$

In particular, there are constants $C_{m_2,1}, C_{m_2,2}$ such that

$$f = -h^2 \left(C_{m_2,1}(x-1)^{4m_2} + C_{m_2,2} E (x-1)^{2m_2-1} \right) (\varpi')^{-4}.$$

From this, we obtain that $|u(x)| \sim |\varphi'|^{-1/2}$ for all x. We localize u by setting

$$\mu = \delta h^{\gamma/(2m_2+1)}, \quad 0 < \delta \ll 1$$

fixing $\chi(s) \in \mathcal{C}_c^\infty(\mathbb{R})$ so that $\chi \equiv 1$ for $|s| \leqslant 1$ and $\operatorname{supp}\chi \subset [-2,2]$, and letting

$$\tilde{u}(x) = \chi((x-1)/\mu) u(x).$$

More calculations, which are again in the spirit of those contained in [CW11], show that $\|\tilde{u}\|_{L^2}^2 \sim h^{(1-2m_2)/(2m_2+3)}$ and

$$(hD)^2 \tilde{u} = (\varpi')^2 \tilde{u} + R,$$

where

$$R = f\tilde{u} + [(hD)^2, \chi((x-1)/\mu)]u.$$

Moreover, the remainder satisfies

$$\|R\|_{L^2} = \mathcal{O}(h^\gamma)\|\tilde{u}\|_{L^2}. \tag{3.1}$$

This quasimode can then be used to saturate the local smoothing estimates near the inflection point. We, again, refer the interested reader to the proof in [CW11, Theorem 3], which provides the following.

Theorem 3. *Let $\varphi_0(x,\theta) = e^{ik\theta}\tilde{u}(x)$, where $\tilde{u} \in \mathcal{C}_c^\infty(\mathbb{R})$ was constructed above. We let $h = |k|^{-1}$, where $|k|$ is taken sufficiently large. Suppose ψ solves*

$$\begin{cases} (D_t - \widetilde{\Delta})\psi = 0, \\ \psi|_{t=0} = \varphi_0. \end{cases}$$

Then for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $\operatorname{supp}\tilde{u}$ and $A > 0$ sufficiently large, independent of k, there exists a constant $C_0 > 0$ independent of k such that

$$\int_0^{|k|^{-4/(2m_2+3)}/A} \| \langle D_\theta \rangle \chi\psi\|_{L^2}^2 dt \geqslant C_0^{-1} \| \langle D_\theta \rangle^{(2m_2+1)/(2m_2+3)} \varphi_0\|_{L^2}^2. \tag{3.2}$$

References

[BGH10] Nicolas Burq, Colin Guillarmou, and Andrew Hassell. Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics. *Geom. Funct. Anal.*, 20(3):627–656, 2010.

[Boo11] Robert Booth. Energy estimates on asymptotically flat surfaces of revolution. *Masters Project, University of North Carolina*, 2011.

[Bur04] N. Burq. Smoothing effect for Schrödinger boundary value problems. *Duke Math. J.*, 123(2):403–427, 2004.

[Chr07] Hans Christianson. Semiclassical non-concentration near hyperbolic orbits. *J. Funct. Anal.*, 246(2):145–195, 2007.

[Chr08] Hans Christianson. Dispersive estimates for manifolds with one trapped orbit. *Comm. Partial Differential Equations*, 33:1147–1174, 2008.

[Chr11] Hans Christianson. Quantum monodromy and nonconcentration near a closed semi-hyperbolic orbit. *Trans. Amer. Math. Soc.*, 363(7):3373–3438, 2011.

[Chr13] Hans Christianson. High-frequency resolvent estimates on asymptotically Euclidean warped products. *preprint*, 2013.

[CKS95] Walter Craig, Thomas Kappeler, and Walter Strauss. Microlocal dispersive smoothing for the Schrödinger equation. *Comm. Pure Appl. Math.*, 48(8):769–860, 1995.

[CPV99] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Distribution of resonances and local energy decay in the transmission problem. II. *Math. Res. Lett.*, 6(3-4):377–396, 1999.

[CPV01] Fernando Cardoso, Georgi Popov, and Georgi Vodev. Asymptotics of the number of resonances in the transmission problem. *Comm. Partial Differential Equations*, 26(9-10):1811–1859, 2001.

[CS88] Peter Constantin and Jean-Claude Saut. Local smoothing properties of dispersive equations. *J. Amer. Math. Soc.*, 1(2):413–439, 1988.

[CV10] Fernando Cardoso and Georgi Vodev. Boundary stabilization of transmission problems. *J. Math. Phys.*, 51(2):023512, 15, 2010.

[CW11] Hans Christianson and Jared Wunsch. Local smoothing for the schrödinger equation with a prescribed loss. *preprint*, 2011.

[Dat09] Kiril Datchev. Local smoothing for scattering manifolds with hyperbolic trapped sets. *Comm. Math. Phys.*, 286(3):837–850, 2009.

[Doi96a] Shin-ichi Doi. Remarks on the Cauchy problem for Schrödinger-type equations. *Comm. Partial Differential Equations*, 21(1-2):163–178, 1996.

[Doi96b] Shin-ichi Doi. Smoothing effects of Schrödinger evolution groups on Riemannian manifolds. *Duke Math. J.*, 82(3):679–706, 1996.

[DV12] Kiril Datchev and András Vasy. Gluing semiclassical resolvent estimates via propagation of singularities. *Int. Math. Res. Not. IMRN*, (23):5409–5443, 2012.

[MMT08] Jeremy Marzuola, Jason Metcalfe, and Daniel Tataru. Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Anal.*, 255(6):1497–1553, 2008.

[RT07] I. Rodnianski and T. Tao. Longtime decay estimates for the Schrödinger equation on manifolds. In *Mathematical aspects of nonlinear dispersive equations*, volume 163 of *Ann. of Math. Stud.*, pages 223–253. Princeton Univ. Press, Princeton, NJ, 2007.

[Sjö87] Per Sjölin. Regularity of solutions to the Schrodinger equation. *Duke Math. J.*, 55(3):699–715, 1987.

[SZ91] Johannes Sjöstrand and Maciej Zworski. Complex scaling and the distribution of scattering poles. *J. Amer. Math. Soc.*, 4(4):729–769, 1991.

[SZ02] Johannes Sjöstrand and Maciej Zworski. Quantum monodromy and semi-classical trace formulae. *J. Math. Pures Appl. (9)*, 81(1):1–33, 2002.

[SZ07] Johannes Sjöstrand and Maciej Zworski. Fractal upper bounds on the density of semiclassical resonances. *Duke Math. J.*, 137(3):381–459, 2007.

[Veg98] Luis Vega. Schrödinger equations: pointwise convergence to the initial data. *Proc. Amer. Math. Soc.*, 102(4):874–878, 1998.

[WZ00] Jared Wunsch and Maciej Zworski. Distribution of resonances for asymptotically Euclidean manifolds. *J. Differential Geom.*, 55(1):43–82, 2000.

E-mail address: hans@math.unc.edu

Department of Mathematics, UNC-Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599

E-mail address: metcalfe@email.unc.edu

Department of Mathematics, UNC-Chapel Hill, CB#3250 Phillips Hall, Chapel Hill, NC 27599