CUTOFF RESOLVENT ESTIMATES AND THE SEMILINEAR SCHRÖDINGER EQUATION

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Abstract. This paper shows how abstract resolvent estimates imply local smoothing for solutions to the Schrödinger equation. If the resolvent estimate has a loss when compared to the optimal, non-trapping estimate, there is a corresponding loss in regularity in the local smoothing estimate. As an application, we apply well-known techniques to obtain well-posedness results for the semi-linear Schrödinger equation.

1. Introduction

In this short note we show how cutoff semiclassical resolvent estimates for the Laplacian on a non-compact manifold, with spectral parameter on the real axis, lead to well-posedness results for the semilinear Schrödinger equation. Motivated by the requirements of [Chr3] and [BGT2], and the microlocal inverse estimates of [Chr1, Chr2], we first prove a general theorem for a large class of resolvents. Following the recent work of Nonnenmacher-Zworski [NoZw], we apply the general theorem in the case there is a hyperbolic fractal trapped set.

Let (M, g) be a Riemannian manifold of dimension n without boundary, with (non-negative) Laplace-Beltrami operator $-\Delta$ acting on functions. The Laplace-Beltrami operator is an unbounded, essentially self-adjoint operator on $L^2(M)$ with domain $H^2(M)$. We assume (M, g) is asymptotically Euclidean in the sense of [NoZw, (3.7)-(3.9)] and that the classical resolvent $(-\Delta - (\lambda^2 + i\epsilon))^{-1}$ obeys a limiting absorption principle as $\epsilon \to 0+$, $\lambda \neq 0$.

Our first result is that if we have cutoff semiclassical resolvent estimates with a sufficiently small loss, then we have weighted smoothing for the Schrödinger propagator with a loss. Let ρ_s be a smooth, non-vanishing weight function satisfying

$$\rho_s(x) \equiv \langle d_g(x, x_0) \rangle^{-s}, \tag{1.1}$$

for some fixed x_0 and x outside a compact set.

Theorem 1. *Suppose for each compactly supported function $\chi \in \mathcal{C}_c^\infty(M)$ with sufficiently small support, there is $h_0 > 0$ such that the semi-classical Laplace-Beltrami operator satisfies*

$$\|\chi(-h^2\Delta - E)^{-1}\chi u\|_{L^2(M)} \leq \frac{g(h)}{h}\|u\|_{L^2(M)}, \quad E > 0 \tag{1.2}$$

uniformly in $0 < h \leq h_0$, where $g(h) \geq c_0 > 0$, $g(h) = o(h^{-1})$. Then for each $T > 0$ and $s > 1/2$, there is a constant $C = C_{T,s} > 0$ such that

$$\int_0^T \left\|\rho_s e^{it\Delta} u_0\right\|^2_{H^{1/2-\eta}(M)} dt \leq C\|u_0\|^2_{L^2(M)}, \tag{1.3}$$

where $\eta \geq 0$ *satisfies*

$$g(h)h^{2\eta} = \mathcal{O}(1), \tag{1.4}$$

and ρ_s *is given by* (1.1).

The assumption that (M, g) is asymptotically Euclidean is that there exists $R_0 > 0$ sufficiently large that, on each infinite branch of $M \setminus B(0, R_0)$, the semiclassical Laplacian $-h^2\Delta$ takes the form

$$-h^2\Delta|_{M\setminus B(0,R_0)} = \sum_{|\alpha|\leq 2} a_\alpha(x,h)(hD_x)^\alpha,$$

with $a_\alpha(x,h)$ independent of h for $|\alpha| = 2$,

$$\sum_{|\alpha|=2} a_\alpha(x,h)(hD_x)^\alpha \geq C^{-1}|\xi|^2, \;\; 0 < C < \infty, \text{ and}$$

$$\sum_{|\alpha|\leq 2} a_\alpha(x,h)(hD_x)^\alpha \to |\xi|^2, \;\; \text{as } |x| \to \infty \text{ uniformly in } h.$$

In order to quote the results of [NoZw] we also need the following analyticity assumption: $\exists \theta_0 \in [0, \pi)$ such that the $a_\alpha(x,h)$ are extend holomorphically to

$$\{r\omega : \omega \in \mathbb{C}^n, \; \mathrm{dist}\,(\omega, \mathbb{S}^n) < \epsilon, \; r \in \mathbb{C}, \; |r| \geq R_0, \; \arg r \in [-\epsilon, \theta_0 + \epsilon)\}.$$

As in [NoZw], the analyticity assumption immediately implies

$$\partial_x^\beta \left(\sum_{|\alpha|\leq 2} a_\alpha(x,h)\xi^\alpha - |\xi|^2 \right) = o(|x|^{-|\beta|}) \langle \xi \rangle^2, \; |x| \to \infty.$$

Recall the free Laplacian $(-\Delta_0 - \lambda^2)^{-1}$ on $\mathbb{R}^n$ has a holomorphic continuation from $\operatorname{Im}\lambda > 0$ to $\lambda \in \mathbb{C}$ for $n \geq 3$ odd, and to the logarithmic covering space for n even. This motivates the limiting absorption assumption, that

$$\lim_{\epsilon \to 0+, \;\; \lambda \neq 0} \rho_s(-\Delta - (\lambda^2 + i\epsilon))^{-1}\rho_s$$

exists as a bounded operator

$$L^2(M, d\mathrm{vol}_g) \to L^2(M, d\mathrm{vol}_g),$$

provided $s > 1/2$. As in the free case, we allow a possible logarithmic singularity at $\lambda = 0$.

The problem of "local smoothing" estimates for the Schrödinger equation has a long history. The sharpest results to date are those of Doi [Doi] and Burq [Bur]. Doi proved if M is asymptotically Euclidean, then one has the estimate

$$\int_0^T \left\| \chi e^{it\Delta} u_0 \right\|^2_{H^{1/2}(M)} dt \leq C \|u_0\|^2_{L^2(M)} \tag{1.5}$$

for $\chi \in \mathcal{C}_c^\infty(M)$ if and only if there are no trapped sets. Burq's paper showed if there is trapping due to the presence of several convex obstacles in $\mathbb{R}^n$ satisfying certain assumptions, then one has the estimate (1.5) with the $H^{1/2}$ norm replaced by $H^{1/2-\eta}$ for $\eta > 0$. In [Chr3], the author considered an arbitary, single trapped hyperbolic orbit. One of the goals of this paper is to use estimates obtained by Nonnenmacher-Zworski [NoZw] for fractal hyperbolic trapped sets to obtain similar results to [Chr3] for the semilinear Schrödinger equation. To that end we have the following corollary to Theorem 1.

Corollary 1.1. *Assume* (M, g) *admits a hyperbolic fractal trapped set,* K_E*, in the energy level* $E > 0$ *and that the topological pressure* $P_E(1/2) < 0$*. Then* $-h^2\Delta - E$ *satisfies* (1.2) *for some* $E > 0$ *with* $g(h) = C\log(1/h)$*, and for every* $\eta > 0$*,* $T > 0$*, and* $s > 1/2$*, there exists a constant* $C = C_{P_E,\eta,T,s} > 0$ *such that*

$$\int_0^T \left\|\rho_s e^{it\Delta}u_0\right\|^2_{H^{1/2-\eta}(M)}\,dt \leq C\|u_0\|^2_{L^2(M)}.$$

We remark that the assumption $P_E(1/2) < 0$ implies the trapped set K_E is filamentary or "thin" (see [NoZw] for definitions).

We consider the following semilinear Schrödinger equation problem:

$$\left\{\begin{array}{c} i\partial_t u + \Delta u = F(u) \text{ on } I \times M; \\ u(0,x) = u_0(x), \end{array}\right. \tag{1.6}$$

where $I \subset \mathbb{R}$ is an interval containing 0. Here the nonlinearity F satisfies

$$F(u) = G'(|u|^2)u,$$

and $G : \mathbb{R} \to \mathbb{R}$ is at least C^3 and satisfies

$$|G^{(k)}(r)| \leq C_k\langle r\rangle^{\beta-k},$$

for some $\beta \geq \frac{1}{2}$.

In §3 we prove a family of Strichartz-type estimates which will result in the following well-posedness theorem.

Theorem 2. *Suppose* (M, g) *satisfies the assumptions of the introduction, and set*

$$\delta = \frac{4\eta}{2\eta+1} \geq 0. \tag{1.7}$$

Then for each

$$s > \frac{n}{2} - \frac{2}{\max\{2\beta-2, 2\}} + \delta \tag{1.8}$$

and each $u_0 \in H^s(M)$ *there exists* $p > \max\{2\beta - 2, 2\}$ *and* $0 < T \leq 1$ *such that* (1.6) *has a unique solution*

$$u \in C([-T, T]; H^s(M)) \cap L^p([-T, T]; L^\infty(M)). \tag{1.9}$$

Moreover, the map $u_0(x) \mapsto u(t, x) \in C([-T, T]; H^s(M))$ *is Lipschitz continuous on bounded sets of* $H^s(M)$*, and if* $\|u_0\|_{H^s}$ *is bounded,* T *is bounded from below.*

If, in addition, (M, g) *satisfies the assumptions of Corollary 1.1,* $n \leq 3$*,* $\beta < 3$*, and* $G(r) \to +\infty$ *as* $r \to +\infty$*, then* u *in* (1.9) *extends to a solution*

$$u \in C((-\infty, \infty); H^1(M)) \cap L^p((-\infty, \infty); L^\infty(M)).$$

Remark 1.2. In particular, the cubic defocusing non-linear Schrödinger equation is globally H^1-well-posed in three dimensions with a fractal trapped hyperbolic set which is sufficiently filamentary. Of course other nonlinearities can be considered, but for simplicity we consider only these in this work.

Acknowledgments. This research was partially conducted during the period the author was employed by the Clay Mathematics Institute as a Liftoff Fellow.

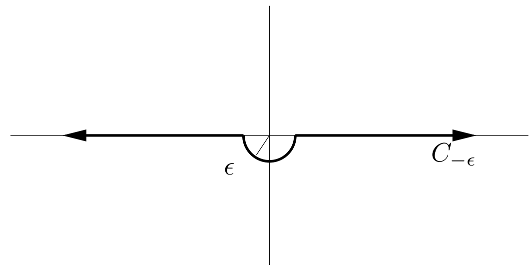

FIGURE 1. The curve $\mathcal{C}_{-\epsilon}$ in the complex plane.

2. Proof of Theorem 1

Since we are assuming $(-\Delta - z)^{-1}$ obeys a limiting absorption principle, we have

$$\|\rho_s(-\Delta - (\tau - i\epsilon))^{-1}\rho_s\|_{L^2 \to L^2} \leq C_\epsilon$$

for $0 < \epsilon_0 \leq |\tau| \leq C$. For $|\sigma| \geq C$ for some $C > 0$, $\sigma \in \mathbb{C}$ in a neighbourhood of the real axis, write

$$\begin{aligned} -\Delta - \sigma &= -\Delta - \frac{z}{h^2} \\ &= h^{-2}(-h^2\Delta - z), \end{aligned}$$

for

$$z \in [E - \alpha, E + \alpha] + i[-c_0 h, c_0 h].$$

Now

$$(-h^2\Delta - z)$$

is a Fredholm operator for z in the specified range, and hence the "gluing" techniques from [Vod] and [Chr3, §2] can be used to conclude for $s > 1/2$,

$$\rho_s(-h^2\Delta - z)^{-1}\rho_s$$

has a holomorphic extension to a slightly smaller neighbourhood in z, and in particular,

$$\|\rho_s(-h^2\Delta - E)^{-1}\rho_s\|_{L^2 \to L^2} \leq C\frac{g(h)}{h}.$$

Rescaling, we have

$$\left\|\rho_s(-\Delta - \tau)^{-1}\rho_s\right\|_{L^2 \to L^2} \leq C\frac{g(\langle\tau\rangle^{1/2})}{\langle\tau\rangle^{1/2}}, \ \ \tau \in \mathcal{C}_{\pm\epsilon}, \tag{2.1}$$

where (see Figure 1)

$$\mathcal{C}_{\pm\epsilon} = \{\tau \in \mathbb{R} : |\tau| \geq \epsilon\} \cup \{\tau \in \mathbb{C} : |\tau| = \epsilon, \pm \operatorname{Im}\tau \geq 0\}.$$

As in [Chr3] and [Bur], the following lemma follows from integration by parts and interpolation, together with the condition on η, (1.4).

Lemma 2.1. *With the notation and assumptions above, we have*

$$\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to H^1} \leq Cg(\langle\tau\rangle^{1/2}), \ \ \tau \in \mathcal{C}_{\pm\epsilon},$$

and for every $r \in [-1, 1]$,

$$\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{H^r \to H^{1+r-\eta/2}} \leq C, \;\; \tau \in \mathcal{C}_{\pm\epsilon}.$$

Theorem 1 now follows from the standard "TT^*" argument, letting $\epsilon \to 0$ in (2.1) (see [BGT2], the references cited therein, and [Chr3]).

□

The following Corollary uses interpolation with an H^2 estimate to replace the $H^{1/2-\eta}$ norm on the left hand side of (1.3) with $H^{1/2}$, and will be of use in §3. See [Chr3] for the details of the proof.

Corollary 2.2. *Suppose* (M, g) *satisfies the assumptions of Theorem 1. For each* $T > 0$ *and* $s > 1/2$*, there is a constant* $C > 0$ *such that*

$$\int_0^T \left\|\rho_s e^{it\Delta} u_0\right\|^2_{H^{1/2}(M)} dt \leq C\|u_0\|^2_{H^\delta(M)}, \tag{2.2}$$

where $\delta \geq 0$ *is given by* (1.7).

In particular, if (M, g) *satisfies the assumptions of Corollary 1.1, then for any* $\delta > 0$*, there is* $C = C_\delta > 0$ *such that* (2.2) *holds.*

3. Strichartz-type Inequalities

In this section we give several families of Strichartz-type inequalities and prove Theorem 2. The statements and proofs are mostly adaptations of similar inequalities in [BGT2], so we leave out the proofs of these in the interest of space.

If we view $M \setminus U$, where U is a neighbourhood of K_E, as a manifold with nontrapping geometry, we may apply the results of [HTW] or [BoTz] to a solution of the Schrödinger equation away from the trapping region, resulting in perfect Strichartz estimates. For this section we need (1.3) only with a compact cutoff χ instead of with the more general weight ρ_s.

Proposition 3.1. *For every* $0 < T \leq 1$ *and each* $\chi \in \mathcal{C}_c^\infty(M)$ *satisfying* $\chi \equiv 1$ *near* U *, there is a constant* $C > 0$ *such that*

$$\|(1-\chi)u\|_{L^p([0,T])W^{s,q}(M)} \leq C\|u_0\|_{H^s(M)}, \tag{3.1}$$

where $u = e^{it\Delta}u_0$*,* $s \in [0, 1]$*, and* (p, q)*,* $p > 2$ *satisfy*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Remark 3.2. In the sequel, wherever unambiguous, we will write

$$L^p_T W^{s,q} := L^p([0,T])W^{s,q}(M)$$

and

$$H^s := H^s(M).$$

Proposition 3.3. *Suppose* (M, g) *satisfies the assumptions of the Introduction,* $u = e^{it\Delta}u_0$*, and*

$$v = \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau.$$

Then for each $0 < T \leq 1$ *and* $\delta \geq 0$ *satisfying* (1.7)*, we have the estimates*

$$\|u\|_{L^p_T W^{s-\delta,q}} \leq C\|u_0\|_{H^s} \tag{3.2}$$

and

$$\|v\|_{L^p_T W^{s-\delta,q}} \leq C\|f\|_{L^1_T H^s}, \tag{3.3}$$

where $s \in [0,1]$ *and* (p,q), $p > 2$ *satisfy the Euclidean scaling*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{3.4}$$

The proof uses a local WKB expansion localized also in time to the scale of inverse frequency, followed by summing over frequency bands (see [Chr3] and [BGT1]). The only difference here is the explicit dependence of δ on η, which is related to the growth of the function $g(h)$.

Proof of Theorem 2. The proof of Theorem 2 is a slight modification of the proof of Proposition 3.1 in [BGT1], but we include it here in the interest of completeness. Fix s satisfying 1.8 and choose $p > \max\{2\beta - 2, 2\}$ satisfying

$$s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{1}{\max\{2\beta - 2, 2\}}$$

where $\delta \geq 0$ satisfies (1.7). Set $\sigma = s - \delta$ and

$$Y_T = C([-T,T]; H^s(M)) \cap L^p([-T,T]; W^{\sigma,q}(M))$$

for

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

equipped with the norm

$$\|u\|_{Y_T} = \max_{|t| \leq T} \|u(t)\|_{H^s(M)} + \|u\|_{L^p_T W^{\sigma,q}}.$$

Let Φ be the nonlinear functional

$$\Phi(u) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\tau)\Delta}F(u(\tau))d\tau.$$

If we can show that $\Phi : Y_T \to Y_T$ and is a contraction on a ball in Y_T centered at 0 for sufficiently small $T > 0$, this will prove the first assertion of the Proposition, along with the Sobolev embedding

$$W^{\sigma,q}(M) \subset L^\infty(M), \tag{3.5}$$

since $\sigma > n/q$. From Proposition 3.3, we bound the W^σ part of the Y_T norm by the H^s norm, giving

$$\begin{aligned} \|\Phi(u)\|_{Y_T} &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^T \|F(u(\tau))\|_{H^s}d\tau\right) \\ &\leq C\left(\|u_0\|_{H^s} + \int_{-T}^T \|(1 + |u(\tau)|)\|_{L^\infty}^{2\beta-2})\|u(\tau)\|_{H^s}d\tau\right), \end{aligned}$$

where the last inequality follows by our assumptions on the structure of F. Applying Hölder's inequality in time with $\tilde{p} = p/(2\beta - 2)$ and $\tilde{q}$ satisfying

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} = 1$$

gives

$$\|\Phi(u)\|_{Y_T} \leq C \left(\|u_0\|_{H^s} + T^\gamma \|u\|_{L^\infty_T H^s} \|(1+|u|)\|^{2\beta-2}_{L^p_T L^\infty}) \right)$$

where $\gamma = 1/\tilde{q} > 0$. Thus

$$\|\Phi(u)\|_{Y_T} \leq C \left(\|u_0\|_{H^s} + T^\gamma (\|u\|_{Y_T} + \|u\|^{2\beta}_{Y_T}) \right).$$

Similarly, we have for $u, v \in Y_T$,

$$\|\Phi(u) - \Phi(v)\|_{Y_T} \leq \tag{3.6}$$

$$\leq CT^\gamma \|u-v\|_{L^\infty_T H^s} \|(1+|u|)\|^{2\beta-2}_{L^p_T L^\infty} + \|(1+|v|)\|^{2\beta-2}_{L^p_T L^\infty}) \tag{3.7}$$

$$\leq CT^\gamma \|u-v\|_{Y_T} \|(1+|u|)\|^{2\beta-2}_{Y_T} + \|(1+|v|)\|^{2\beta-2}_{Y_T}),$$

which is a contraction for sufficiently small T. This concludes the proof of the first assertion in the Proposition.

To get the second assertion, we observe from 3.6 and the definition of Y_T, if u and v are two solutions to (1.6) with initial data u_0 and u_1 respectively, so

$$\widetilde{\Phi}(v) = e^{it\Delta} u_1 - i \int_0^t e^{i(t-\tau)\Delta} F(v(\tau)) d\tau,$$

we have

$$\begin{aligned} &\max_{|t|\leq T} \|u(t) - v(t)\|_{H^s} \\ &= \max_{|t|\leq T} \|\Phi(u)(t) - \widetilde{\Phi}(v)(t)\|_{H^s} \\ &\leq C \Bigg(\|u_0 - u_1\|_{H^s} \\ &\quad + T^\gamma \max_{|t|\leq T} \|u(t)-v(t)\|_{H^s} \|(1+|u|)\|^{2\beta-2}_{L^p_T L^\infty} + \|(1+|v|)\|^{2\beta-2}_{L^p_T L^\infty}) \Bigg), \end{aligned}$$

which, for $T > 0$ sufficiently small gives the Lipschitz continuity.

If (M, g) satisfies the assumptions of Corollary 1.1, $n \leq 3$, $\beta < 3$, and $G(r) \to +\infty$ as $r \to +\infty$, we can take s and p satisfying $p > \max\{2\beta - 2, 2\}$ and

$$s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{2}{\max\{2\beta-2, 2\}}$$

for any $\delta > 0$. Then $\sigma = s - \delta > q/n$ and the preceding argument holds. Finally, the proof of the global well-posedness now follows from the standard global well-posedness arguments from, for example, [Caz, Chapter 6]. □

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