FROM RESOLVENT ESTIMATES TO DAMPED WAVES

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ABSTRACT. In this paper we show how to obtain decay estimates for the damped wave equation on a compact manifold without geometric control via knowledge of the dynamics near the un-damped set. We show that if replacing the damping term with a higher-order *complex absorbing potential* gives an operator enjoying polynomial resolvent bounds on the real axis, then the "resolvent" associated to our damped problem enjoys bounds of the same order. It is known that the necessary estimates with complex absorbing potential can also be obtained via gluing from estimates for corresponding non-compact models.

1. INTRODUCTION

On a compact, connected Riemannian manifold without boundary (X, g), we consider the non-selfadjoint Schrödinger operator

$$P(h) = h^2\Delta_g + \mathrm{i}\, ha \tag{1.1}$$

where $a \in C^\infty(X)$ is a non-negative function, and $\Delta_g = d^*d$ is the *non-negative* Laplacian associated to the metric g. This paper mainly addresses the question of the semiclassical analysis of the resolvent of $P(h)$,

$$R_z(h) := (P(h) - z)^{-1}$$

for z in a complex $h-$dependent neighborhood of 1. For non-selfadjoint operators, it is well known that the norm of the resolvent $\|R_z(h)\|_{\mathcal{L}(L^2,L^2)}$ may be large, even far from the spectrum [18], and a better understanding of the resolvent properties of non-selfadjoint operators remains a challenging problem [30]. In this paper we are particularly interested in (polynomial) upper bounds in h for the resolvent. These bounds are especially useful when studying the stabilization problem, which deals with the rate of the energy decay of the solution of the *damped wave equation* on X :

$$\begin{cases} \left(\partial_t^2 + \Delta_g + a(x)\partial_t\right) u(x,t) = 0, & (x,t) \in X \times (0,\infty) \\ u(x,0) = u_0 \in H^1(X), \quad \partial_t u(x,0) = u_1 \in H^0(X). \end{cases} \tag{1.2}$$

It has been shown (see [23]) that if $a > 0$ somewhere, then the energy of the waves,

$$E(u,t) = \frac{1}{2}\int_X |\partial_t u|^2 + |\nabla u|^2 dx$$

satisfies $E(u,t) \xrightarrow{t\to\infty} 0$ for any initial data $(u_0, u_1) \in H^1 \times H^0$. If some monotone decreasing function $f(t)$ can be found such that

$$E(u,t) \leqslant f(t)E(u,0)\,,$$

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so-called strong stabilization occurs. It is not hard to show that this is equivalent to a uniform exponential decay : $\exists C, \beta > 0$ such that for any u solution of (1.2),

$$E(u,t) \leqslant Ce^{-\beta t} E(u,0).$$

In pioneering works of Rauch, Taylor, Bardos and Lebeau [2, 23, 26], it has been shown in various settings that strong stabilization is equivalent to the geometric control condition (GCC) : there exists $T_0 > 0$ such that from every point in $\Sigma = \{|\xi|_g^2 = 1\} \subset T^*X$, the bicharacteristic of $P(h)$ reaches $\{a > 0\}$ in time $\leqslant T_0$. By contrast, when the manifold X is no longer controlled by a, decay rate estimates usually involve additional regularity of the initial data. They take the form

$$E(u,t) \leqslant f_s(t) \|u\|^2_{\mathcal{H}^s}$$

for $s > 0$ and

$$\|u\|_{\mathcal{H}^s} = \|u(0)\|^2_{H^{1+s}} + \|\partial_t u(0)\|^2_{H^s}.$$

The question of exponential energy decay reduces to the study of high-frequency phenomena, in particular the issue of the spectral properties in the semiclassical limit $h \to 0$ of certain non-selfadjoint operators approximately of the form* (1.1), on a fixed energy layer. For instance, when geometric control holds, there exist $h_0 > 0$, $C, c > 0$ such that for $z \in [1-\delta, 1+\delta] + \mathrm{i}[-ch, ch]$ we have

$$\|R_z(h)\|_{\mathcal{L}(L^2,L^2)} \leqslant C/h, \;\; h < h_0. \tag{1.3}$$

Standard arguments then show that this resolvent estimate implies the uniform exponential decay for the energy. Similar arguments will apply in the case considered here, of resolvent estimates with loss.

1.1. **Motivation.** While our main motivation for studying resolvent estimates for $P(h)$ come from the stabilization problem, our approach in this paper is oriented by geometric considerations, as we explain now. As discussed above, in the presence of geometric control, the resolvent $(P(h) - z)^{-1}$ enjoys a polynomial bound in a neighborhood of size ch around the real axis, and this property implies exponential damping. When geometric control no longer holds, it is then a natural question to ask what type of estimate can be satisfied by $\|(P(h) - z)^{-1}\|$, and, crucially, in what type of complex neighborhood of the real axis a resolvent estimate can be obtained. The properties of the undamped set

$$\mathcal{N} = \{\rho \in S^*X : \forall t \in \mathbb{R}, a \circ \mathrm{e}^{t\mathsf{H}_p}(\rho) = 0\} \tag{1.4}$$

are of central importance for this question. Here, H_p denotes the Hamiltonian vector field generated by the principal symbol $p = \sigma_h(P(h)) = |\xi|_g^2$ of the operator $P(h)$, and $S^*X = p^{-1}(1)$ denotes the unit cosphere bundle. We remind the reader that the flow generated by H_p in S^*X is simply the geodesic flow.

We now review some known results in the case $\mathcal{N} \neq \emptyset$. In [7], the case when $\mathcal{N}$ is a single hyperbolic orbit is analyzed, and a polynomial resolvent estimate for $\|(P - z)^{-1}\|$ is shown in a $h/|\log h|$-size neighborhood of the real axis. As a consequence, the energy decay is sub-exponential : with the above notation, one can get $f_s(t) = \mathrm{e}^{-\beta_s \sqrt{t}}$. It is known from recent work [3] that this decay is sharp. If

*Strictly speaking, in order to apply resolvent bounds to the damped wave equation, we also need the imaginary part of the Schrödinger operator to be mildly z dependent, with

$$R_z(h) = (h^2\Delta_g + \mathrm{i}\, ha\sqrt{z} - z)^{-1};$$

this will be handled by perturbation (see Corollary 4.3, Section 6, as well as references [9, 21, 23]).

the curvature of X is assumed to be negative, and if the relative size of the damping function a is sufficiently large, then the resolvent obeys a polynomial bound in a size h neighborhood of the real axis, and as a result, exponential decay for regular initial data occurs [28, 29]. Indeed, the hypotheses in [29] is much more general, requiring only undamped sets of small pressure; the hyperbolic geodesic is a special case. We note also that for *constant* negative curvature, the need for an arbitrarily large damping function a has been recently removed by Nonnenmacher, by using different methods [24].

A natural question raised by the above remarks, is the following: to what extent does the geometry of the trapped set *alone* determine a type of decay? In other words, given a trapped geometry, what type of resolvent estimate do we expect, and in what complex neighborhood of the real axis? This amounts in many cases to a potentially rather crude decay rate for the energy, as it only depends on the structure of $\mathcal{N}$ and not on the global dynamics of geodesics passing through the damping; in certain cases, however, our results can be seen to be sharp.†

Motivated by the "black box" approach of Burq-Zworski [4] (cf. earlier work of Sjöstrand-Zworski [32]) as well as recent work on the gluing of resolvent estimates by Datchev-Vasy [15], we give a recipe for taking information from *resolvent* estimates obtained for a noncompact problem in which the set $\mathcal{N}$ consists of all trapped geodesics—those not escaping to infinity—and investigate what these estimates imply for the compact problem with damping. In practice, as recent results of Datchev-Vasy [15] have shown the resolvent estimates on manifolds with, say, asymptotically Euclidean ends to be equivalent to estimates on a compact manifold with a *complex absorbing potential* substituting for the noncompact ends, we choose for the sake of brevity and elegance to take this latter model as our "noncompact" setting. As will be discussed below, these complex absorbing potentials have the effect of annihilating semiclassical wavefront set along geodesics passing through them; this is why they are roughly interchangeable with noncompact ends, along which energy can flow off to infinity never to return.

We thus formulate our main question as follows. Assuming that (X, g) and a are given, we consider a model operator of the form

$$P_1(h) = h^2\Delta_g + \mathrm{i}\, W_1$$

in which the damping is replaced by a *complex absorbing potential.* We assume that this model operator enjoys a given "resolvent" estimate‡ on the real axis:

$$\|(P_1(h) - z)^{-1}\| \leqslant C\frac{\alpha(h)}{h}, \quad z \in [1-\delta, 1+\delta]. \tag{1.5}$$

Various examples of such estimates already appear in the literature, see for instance [5,7,8,11,12,25,35,37] and the references therein. Given (1.5) we then aim to obtain analogous estimates for the inverse of the operator with damping, i.e., on

$$(P(h) - z)^{-1} = (h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}\,,$$

when the complex absorbing potential has crucially been replaced by an $O(h)$ damping term. In this paper, we address this question using a control theory argument

†A very natural further question would then be : given a trapped geometry, what kind of global assumptions on the manifold can improve the crude decay rate obtained when only $\mathcal{N}$ is known?

‡We refer to this as a resolvent estimate owing to its close relationship with the estimate for the resolvent in scattering problems.

motivated by [4], together with a recently improved estimate on resolvents truncated away from the trapped set on one side [17], and show that we obtain the same order of estimate as for the model operator. In the next subsection we state the precise results.

1.2. **Results.** As above, we take H_p to be the Hamilton vector field of $p = |\xi|_g^2 - 1$ and $\mathrm{e}^{t\mathsf{H}_p}$ its bicharacteristic flow inside $p^{-1}(1) = S^*X$ (i.e., geodesic flow). We continue to take

$$\mathcal{N} = \{\rho \in S^*X : \forall t \in \mathbb{R}, a \circ \mathrm{e}^{t\mathsf{H}_p}(\rho) = 0\}, \tag{1.6}$$

and will add the further assumption that

$$\overline{\pi(\mathcal{N})} \cap \operatorname{supp} a = \emptyset, \tag{1.7}$$

where π is projection $T^*X \to X$. Thus there exists a non-empty open set O_1 such that $\operatorname{supp}(a) \subset X \setminus O_1$, and $\pi(\mathcal{N}) \Subset O_1$. The following Theorem is our main "black box" spectral estimate.

Theorem 1.1. *Assume that for some $\delta \in (0,1)$ fixed and $K \in \mathbb{Z}$, there is a function $1 \leqslant \alpha(h) = \mathcal{O}(h^{-K})$ such that*

$$\|(h^2\Delta_g + \mathrm{i}\, a - z)^{-1}\|_{L^2 \to L^2} \leqslant \frac{\alpha(h)}{h},$$

for $z \in [1-\delta, 1+\delta]$. Then there exists $C, c_0 > 0$ such that

$$\|(h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}\|_{L^2 \to L^2} \leqslant C\frac{\alpha(h)}{h},$$

for $z \in [1-\delta, 1+\delta] + \mathrm{i}[-c_0, c_0]h/\alpha(h)$.

When $\mathcal{N} = \emptyset$, one has $\alpha(h) = \mathcal{O}(1)$, while for $\mathcal{N} \neq \emptyset$, one has $\alpha(h) \to \infty$ as $h \to 0$. As a general heuristic, the "larger" the trapped set is, the larger is $\alpha(h)$ when $h \to 0$, and the weaker the above global estimate is—see section 5 below for examples.

Remark 1.2. As discussed above, instead of the assumption on the model operator $h^2\Delta_g + \mathrm{i}\, W_1$ with complex absorption, we could just as well, by the results of [15], have made an assumption on a model operator in which the set O_1 is "glued" to non-compact ends of various forms. In particular, it would suffice to know the cut-off resolvent estimate on the real axis for the limiting resolvent

$$\left\|\chi(h^2\Delta' - z + i0)^{-1}\chi\right\|_{\mathcal{L}(L^2,L^2)} \leqslant \frac{\alpha(h)}{h}$$

for a localizer χ equal to 1 on O_1 and for Δ' the Laplacian on a manifold with Euclidean ends whose trapped set is contained in a set O_1' isometric to O_1.

Remark 1.3. The hypotheses of Theorem 1.1 can be weakened to *phase space* hypotheses, with a a pseudodifferential operator (as in [31]). We have chosen to keep the damping as a function on the base in accordance with tradition and for the sake of brevity.

Remark 1.4. The assumption that $\alpha(h) = \mathcal{O}(h^{-K})$ is of a technical nature. It does not appear to be too restrictive, however, since every *known* estimate for weakly unstable trapping satisfies this assumption (see Section 5 below). Indeed, if the undamped set $\mathcal{N}$ is at least weakly semi-unstable, the results of [11] suggest that in fact $\alpha(h)$ is always $\mathcal{O}_\epsilon(h^{-1-\epsilon})$ for any $\epsilon > 0$.

Remark 1.5. If $\alpha(h)$ is not of polynomial nature, the proof of Theorem 1.1 has to be slightly modified (see below). As a result, the final estimate we can obtain is weaker : there exists $C, c_0 > 0$ such that

$$\|(h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}\|_{L^2\to L^2} \leqslant C\frac{\alpha^2(h)}{h},$$

for $z \in [1-\delta, 1+\delta] + \mathrm{i}[-c_0, c_0]h/\alpha^2(h)$.

In section 5 we describe three different settings in which our gluing results apply, in which the dynamics in a neighborhood of the trapped set are respectively

(1) Normally hyperbolic
(2) Degenerate hyperbolic
(3) Hyperbolic with a condition on topological pressure.

In addition to proving resolvent estimates in these settings, we discuss applications to decay rates for solutions to the damped wave equation.

2. Operators with complex absorbing potentials

In this section, we collect some standard results about operators with complex absorbing potentials. Such a potential has a much stronger effect than damping, namely (in the microlocal absence of forcing), that of annihilating semiclassical wavefront set completely along bicharacteristics passing through it in the forward direction.

We collect basic results about the resolvent of an operator with complex absorbing potential. This includes both existence of the family and the basic propagation estimates, which tell us that the complex absorbing potential kills off wavefront set under forward propagation. We begin with the definition of the "resolvent:"

Lemma 2.1. *Let $W \in C^\infty(X)$, $W \geqslant 0$ and W not identically zero. Suppose that*

$$P_1 = h^2\Delta_g + \mathrm{i}\, W$$

on X. Then $(P_1 - z)^{-1}$ is a meromorphic family of bounded operators on L^2 for all $z \in \mathbb{C}$, analytic in the closed lower half-plane.

Proof. We simply remark that $h^2\Delta + 1$ is invertible, and apply the analytic Fredholm theorem to conclude that there exists a meromorphic resolvent family. By the Fredholm alternative, any pole has to correspond to nullspace of $P_1 - z$. Since

$$\operatorname{Im}\langle P_1 u, u\rangle = \langle Wu, u\rangle - (\operatorname{Im} z)\|u\|^2,$$

for u to be in the nullspace would imply that $\operatorname{Im} z \geqslant 0$; equality would further require that $\langle Wu, u\rangle = 0$ which is forbidden by unique continuation. □

Finally, we recall the microlocal bound of propagation through trapping by Datchev-Vasy [16], as well as basic backward propagation of singularities in the presence of complex absorption (see also Lemma A.2 of [25] for the latter). In this context we will say that a bicharacteristic γ (by bicharacteristic we always mean an integral curve of $H_{\operatorname{Re} p_1}$ in the characteristic set of $\operatorname{Re} p_1 - \operatorname{Re} z$ where p_1 is the semiclassical principal symbol of P_1), or a point on γ, is *non-trapped* if $W(\gamma(T)) > 0$ for some $T \in \mathbb{R}$, and is *trapped* otherwise. We say it is *forward non-trapped* if $W(\gamma(T)) > 0$ for some $T > 0$. In the terminology of [16] we say that[§] the resolvent

[§]We have opposite signs for imaginary parts of p_1 relative to [16], so incoming and outgoing are interchanged.

$(P_1 - z)^{-1}$ is *semiclassically incoming with a loss of* h^{-1} provided that whenever $q \in T^*X$ is on a forward non-trapped bicharacteristic γ of P_1 and $f = \mathcal{O}(1)$ on $\gamma|_{[0,T]}$ then $(P_1 - z)^{-1} f$ is $\mathcal{O}(h^{-1})$ at q. To be more precise, this means that for $\psi \in C_c^\infty(T^*X)$ with sufficiently small support near q, we have an inequality

$$\|\psi^w (P_1 - z)^{-1} f\| \leqslant C h^{-1} \|f\|$$

for some $C > 0$.

Lemma 2.2. *(See [16, Theorem 1.3, Lemma 6.1].) Suppose that* $P_1 = h^2\Delta_g + \mathrm{i}\, W$ *on* X, $W \geqslant 0$, *and* $z \in \mathbb{C}$ *such that* $\operatorname{Im} z = \mathcal{O}(h^\infty)$, $\operatorname{Re} z \in [1-\delta, 1+\delta]$ *for* $0 < \delta < 1$ *fixed. Assume that the resolvent is polynomially bounded, i.e.,*

$$\left\|(P_1 - z)^{-1}\right\|_{\mathcal{L}(L^2,L^2)} \leqslant C h^{-K} \quad \text{for some } K.$$

Then $(P_1 - z)^{-1}$ *is semiclassically incoming with a loss of* h^{-1}. *In particular, if* $W(\gamma(T)) > 0$ *for some* $T > 0$ *and* $\mathrm{WF}_\hbar(f)$ *is disjoint from* $\gamma|_{[0,T]}$, *then* $(P_1 - z)^{-1} f = \mathcal{O}(h^\infty)$ *at* q.

Further, if $\chi \in C^\infty(X)$ *and* $T^*_{\operatorname{supp}\chi} X$ *contains no trapped points, then* $\chi(P_1 - z)^{-1}\chi$ *is* $\mathcal{O}(h^{-1})$.

A further result that is of crucial importance in avoiding losses in our estimates is the following result of [17]:

Lemma 2.3. *(See [17, Theorem 2].) With the notation of Lemma 2.2, if*

$$\left\|(P_1 - z)^{-1}\right\|_{\mathcal{L}(L^2,L^2)} \leqslant \frac{\alpha(h)}{h}$$

and if $\chi \in C^\infty(X)$ *and* $T^*_{\operatorname{supp}\chi} X$ *contains no trapped points, then for some* $C > 0$,

$$\left\|(P_1 - z)^{-1}\chi\right\|_{\mathcal{L}(L^2,L^2)} \leqslant C \frac{\sqrt{\alpha(h)}}{h}.$$

3. Propagation and damping estimates

We now switch from complex absorbing potentials back to damping: set

$$P = P(h) = h^2 \Delta_g + \mathrm{i}\, h a$$

and consider the equation

$$(P - z) u = f\,, \quad z \in [1 - \delta, 1 + \delta]\,.$$

We also set $\Sigma = p^{-1}(1) \subset T^*X$ where $p = \sigma_h(P)$. Let us start by recalling a classical result about propagation estimates (see, e.g. Theorem 12.5 of [38] for a proof by conjugation to normal form; an alternative is the usual commutator argument as described in [22] in the homogeneous setting and [8] in the semiclassical setting):

Lemma 3.1. *Suppose* $q \in \Sigma$ *and for some* $T > 0$ *the forward bicharacteristic* $\exp([0,T]\mathsf{H}_p)(q)$ *is disjoint from a compact set* K. *Then there are* Q, Q' *which are elliptic at* q, *resp.* $\exp(T\mathsf{H}_p)(q)$, *with* $\mathrm{WF}'_\hbar(Q') \cap K = \emptyset$ *such that* $\|Qu\| \leqslant \|Q'u\| + Ch^{-1}\|f\|$, $u = (P - z)^{-1} f$.

4. Proof of Theorem 1.1

For the moment, we consider only the case where $z \in [1-\delta, 1+\delta]$. We have

$$P = h^2\Delta_g + \mathrm{i}\, ha$$

and we additionally write

$$P_1 = h^2\Delta_g + \mathrm{i}\, a$$

for the operator with damping replaced by absorption. Choose an open set V_1 such that $\pi(\mathcal{N}) \Subset V_1 \Subset O_1$. Let $B_1, \varphi \in C_0^\infty(X)$ be smooth functions with $B_1|_{V_1} = 1$, $\operatorname{supp} B_1 \subset O_1$, $\varphi = 1$ on $\operatorname{supp} \nabla B_1$ and $\operatorname{supp}\varphi \cap \pi(\mathcal{N}) = \emptyset$. We observe that $\pi(\mathcal{N})$ satisfies the assumptions of Lemma 2.3, so that

$$\|(P_1 - z)^{-1}\varphi u\| \leqslant Ch^{-1}\sqrt{\alpha(h)}\|\varphi u\|.$$

Then, noticing that a and B_1 have disjoint supports, we have

$$\begin{aligned}
\|B_1 u\| &= \|(P_1 - z)^{-1}(P_1 - z)B_1 u\| \\
&= \|(P_1 - z)^{-1}(P - z)B_1 u\| \\
&= \|(P_1 - z)^{-1}(B_1(P - z) + [P, B_1])u\| \\
&\leqslant \|(P_1 - z)^{-1}B_1(P - z)u\| + \|(P_1 - z)^{-1}\varphi[P, B_1]\varphi u\| \\
&\leqslant \frac{\alpha(h)}{h}\|B_1(P - z)u\| + C\sqrt{\alpha(h)}\|\varphi u\|.
\end{aligned} \tag{4.1}$$

To get the second term in the last line, we have implicitly used standard energy cutoffs and ellipticity of $P - z$ away from the characteristic variety to control the growth in ξ of the commutator $[P, B_1]$. See, for instance, the arguments in [8].

Since for $\rho \notin \mathcal{N}$ the curve $e^{t\mathsf{H}_p}(\rho)$ passes through $\{a > 0\}$, each such bicharacteristic curve must certainly enter the compact set $X \backslash O_1$. Thus by compactness, there exists $\epsilon_0 > 0$ such that every such curve passes through $\{a \geqslant \epsilon_0\}$. We now take a cutoff function $\chi \geqslant 0$ with $\operatorname{supp}\chi \subset \{a > \epsilon_0/2\}$, and $\chi = 1$ whenever $a \geqslant \epsilon_0$; hence every controlled geodesic passes through $\{\chi = 1\}$.

We next recall a classical lemma concerning the propagation of singularities in the presence of geometric control. (See [4], which builds on a semiclassical version of [22], proved in [33].) This is a slight variation on Lemma 3.1, and can of course also be proved using the original positive commutator argument (see [8]).

Lemma 4.1. *(See for instance [4, Lemma 4.1].) Let U be an open neighborhood of $\mathcal{N}$, $\chi \in C^\infty(X)$ as above. If $B_0 = \Psi_h^{0,0}(X)$ is such that $\mathrm{WF}'_h(B_0) \subset T^*X \setminus U$, then for z real near 1,*

$$\|B_0 u\| \leqslant \frac{C}{h}\|(P - z)u\| + \|\chi u\| + \mathcal{O}(h^\infty)\|u\|$$

Since $\operatorname{supp}\varphi$ and $\operatorname{supp}(1 - B_1)$ lie inside the controlled region, we can write :

$$\|(I - B_1)u\| \leqslant Ch^{-1}\|(P - z)u\| + C\|\chi u\| + \mathcal{O}(h^\infty)\|u\|$$

and

$$\|\varphi u\| \leqslant Ch^{-1}\|(P - z)u\| + C\|\chi u\| + \mathcal{O}(h^\infty)\|u\|\,.$$

But

$$\begin{aligned}\|\chi u\|^2 &\leqslant C\langle au, u\rangle \\ &= Ch^{-1}\operatorname{Im}\langle (P-z)u, u\rangle \\ &\leqslant Ch^{-1}\|(P-z)u\|\|u\|.\end{aligned}$$

Starting with (4.1), we deduce from the above inequalities that

$$\begin{aligned}\|B_1 u\|^2 &\leqslant C\Big(\frac{\alpha^2(h)}{h^2}\|(P-z)u\|^2 + C\alpha(h)\|\varphi u\|^2\Big) \\ &\leqslant C\Big(\frac{\alpha^2(h)}{h^2}\|(P-z)u\|^2 + C\alpha(h)\|\chi u\|^2 + \mathcal{O}(h^\infty)\|u\|^2\Big).\end{aligned}$$

Hence, we have

$$\begin{aligned}\|u\|^2 &\leqslant C(\|B_1 u\|^2 + \|(I-B_1)u\|^2) \\ &\leqslant C\Big(\frac{\alpha^2(h)}{h^2}\|(P-z)u\|^2 + \alpha(h)\|\chi u\|^2 + \mathcal{O}(h^\infty)\|u\|^2\Big) \\ &\leqslant C\Big(\frac{\alpha^2(h)}{h^2}\|(P-z)u\|^2 + \frac{\alpha(h)}{h}\|(P-z)u\|\|u\| \\ &\quad + \mathcal{O}(h^\infty)\|u\|^2\Big) \\ &\leqslant C\Big(\frac{\alpha^2(h)}{h^2}\|(P-z)u\|^2 + \frac{4\epsilon^{-1}\alpha^2(h)}{h^2}\|(P-z)u\|^2 + \epsilon\|u\|^2 \\ &\quad + \mathcal{O}(h^\infty)\|u\|^2\Big).\end{aligned}$$

If ϵ is small, we can absorb the last two terms in the above inequality on the left hand side, and we obtain

$$\|u\| \leqslant C\frac{\alpha(h)}{h}\|(P-z)u\|.$$

Now simply observe that by the triangle inequality this bound is still valid if we add to z an imaginary part that satisfies

$$|\operatorname{Im} z| \leqslant h\alpha^{-1}(h)C'$$

for C' such that $C'C < 1$, and this concludes the proof of the theorem.

Remark 4.2. If $\alpha(h)$ is not of polynomial nature, then Lemma 2.3 cannot be used. As a result, the square root in Equation (4.1) must be removed. The rest of the argument is the same, and we end up with the estimate given in Remark 1.5. Note also that the energy decay rates for the damped wave equation are of course weaker than in the case where Lemma 2.3 can be applied.

In order to apply Theorem 1.1 to the situation of the stationary damped wave operator, we now state a corollary where the Schrödinger operator depends mildly on z.

Corollary 4.3. *Let X be a compact manifold without boundary, and let $\widetilde{P}(h,z)$ be the modified operator*

$$\widetilde{P}(h,z) = h^2\Delta_g + \mathrm{i}\, h\sqrt{z}a - z\,.$$

Assume that for some $\delta \in (0,1)$ fixed there is a function $1 \leqslant \alpha(h) = \mathcal{O}(h^{-K})$, for some $K \in \mathbb{Z}$, such that

$$\|(h^2\Delta_g + \mathrm{i}\, a - z)^{-1}\|_{L^2\to L^2} \leqslant \frac{\alpha(h)}{h},$$

for $z \in [1-\delta, 1+\delta]$. Then there exists $C, c_0 > 0$ such that

$$\|\widetilde{P}(h,z)^{-1}\|_{L^2\to L^2} \leqslant C\frac{\alpha(h)}{h},$$

for

$$z \in [1 - c_0/\alpha(h), 1 + c_0/\alpha(h)] + \mathrm{i}[-c_0, c_0]h/\alpha(h).$$

Proof. The real part of the perturbation is manifestly bounded by a small multiple of $h/\alpha(h)$ and can thus be perturbed away by Neumann series. It thus only remains to check that the size of the imaginary part of the perturbation:

$$\mathrm{Re}(ha - \sqrt{z}ha) - \mathrm{Im}\, z$$

can also be made much smaller than $h/\alpha(h)$.

Take $\sqrt{z} = 1+r+\mathrm{i}\,\beta$ for r, β to be determined. Then $z = (1+r)^2-\beta^2+2(1+r)\,\mathrm{i}\,\beta$. Then

$$\mathrm{Re}(ha - \sqrt{z}ha) - \mathrm{Im}\, z = ha(1-(1+r)) - 2(1+r)\beta = \epsilon h/\alpha(h)$$

for $\epsilon > 0$ small if, say, $|\beta| \leqslant \epsilon h/2\alpha(h)$ and $|r| \leqslant \alpha^{-1}(h)$. Squaring, we obtain

$$z \in [1 - c_0/\alpha(h), 1 + c_0/\alpha(h)] + \mathrm{i}[-c_0, c_0]h/\alpha(h).$$

□

5. Examples

In this section, we briefly outline some known microlocal resolvent estimates, state the two different stationary damped wave operator estimates, and then draw conclusions about solutions to the damped wave equation (1.2).

5.1. A normally hyperbolic trapped set.

In this section, we treat the case in which the trapped set is a smooth manifold in S^*X around which the dynamics is *normally hyperbolic*. In this case, estimates for the resolvent with a complex absorbing potential have been obtained by Wunsch-Zworski [37]. A particular case of interest is the "photon sphere" for the Kerr black hole geometry, where the phase space is 6-dimensional $\mathcal{N}$ is a symplectic submanifold, diffeomorphic to T^*S^2—see section 2 of [37] for details on this application, and [34] for placing it in actual Kerr-de Sitter space. Another special case is of course that of a single hyperbolic closed geodesic (discussed further in the following section).

The precise formulation of normal hyperbolicity used here is as follows: we define the backward-forward trapped sets by

$$\Gamma_\pm = \{\rho \in T^*X : \forall t \gtrless 0, a \circ \mathrm{e}^{t\mathsf{H}_p}(\rho) = 0\}$$

Then of course

$$\mathcal{N} = \Gamma_+ \cap \Gamma_-,$$

where we have now ceased to restrict to a single energy surface (so $\mathcal{N} \subset T^*X$ is a homogeneous subset in view of the homogeneity of p) in order to employ the terminology of symplectic geometry more easily.

We make the following assumptions on this intersection:

(1) $\Gamma_\pm$ are codimension-one smooth manifolds intersecting transversely at $\mathcal{N}$. (It is not difficult to verify that $\Gamma_\pm$ must then be coisotropic and $\mathcal{N}$ symplectic.)

(2) The flow is hyperbolic in the normal directions to $\mathcal{N}$ within the energy surface S^*X: there exist subbundles $E^\pm$ of $T_{\mathcal{N}}(\Gamma_\pm)$ such that at $p \in S^*X$

$$T_{\mathcal{N}}\Gamma_\pm = T\mathcal{N} \oplus E^\pm,$$

where

$$d(\exp(\mathsf{H}_p) : E^\pm \to E^\pm$$

and there exists $\theta > 0$ such that

$$\|d(\exp(\mathsf{H}_p)(v)\| \leqslant Ce^{-\theta|t|}\|v\| \text{ for all } v \in E^\mp,\ \pm t \geqslant 0. \tag{5.1}$$

As discussed in [37], these hypotheses as stated are not structurally stable, but they do follow (at least up to loss of derivatives) from the more stringent hypothesis that the dynamics be *r-normally-hyperbolic* for every r in the sense of [20, Definition 4]. This implication, and the structural stability of the hypotheses, follows from a deep theorem of Hirsch-Pugh-Shub [20] and Fenichel [19].

As a consequence of the estimates of [37] for resolvents, we then obtain the following estimate for the damped operator:

Theorem 5.1. *Let (X, g) satisfy the dynamical conditions enumerated above. Then we have*

$$\|(h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}\|_{L^2\to L^2} \leqslant C\frac{|\log h|}{h},$$

for $z \in [1-\delta, 1+\delta] + \mathrm{i}[-c_0, c_0]h/|\log h|$.

This estimate, or more precisely its refinement in Corollary 4.3, provide a corresponding energy decay estimate for solutions to the damped wave equation (1.2). In order to avoid irritating issues of projecting away from constant subspaces, etc., we assume that $u_0 \equiv u(x,0) = 0$.

Corollary 5.2. *Assume the hypotheses of Theorem 5.1 hold, and let u be a solution to (1.2) with $u_0 = 0$, and $u_1 \in H^s$ for some $s \in (0,2]$. Then there exists a constant $C = C_s > 0$ such that*

$$E(u,t) \leqslant Ce^{-st^{1/2}/C}\|u_1\|^2_{H^s}.$$

A simple situation in which the hypotheses of Theorem 1.1 are satisfied is that of a connected compact manifold of the form $X = X_0 \cup X_1$ with X_1 open and X_0 isometric to a warped product $\mathbb{R}_u \times S^{n-1}_\theta$ with metric

$$g = du^2 + \cosh^2 u\, d\theta^2.$$

We take $a \in C^\infty(X)$ to be identically 1 on X_1 as well as equal to 1 for $|x| > 1$ in X_0. This class of manifolds thus includes the "peanut of rotation" shown in Figure 1 as well as its higher dimensional generalizations.

Then the trapped set is easily seen to be $\mathcal{N} = \{u = 0, \xi = 0\}$ where ξ is the cotangent variable dual to u, and the function $x = 2 - u^2$ satisfies the convexity hypotheses. The flow on $\mathcal{N}$ is normally hyperbolic, with the stable and unstable manifolds being the two components of the set

$$(\xi^2 + \frac{|\eta|^2_{S^{n-1}}}{\cosh^2 u}) = |\eta|^2_{S^{n-1}}$$

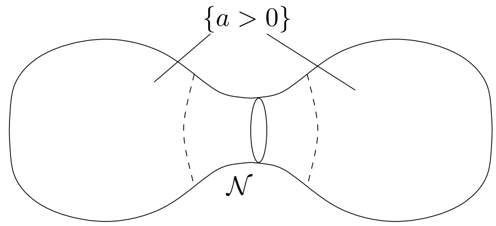

FIGURE 1. The "peanut of rotation".

i.e., by the intersection of the condition that energy and angular momentum match their values on $\mathcal{N}$. This is an example of a normally hyperbolic trapped set, and hence both parts of Theorem 5.1 apply.

5.2. **A trapped set with degenerate hyperbolicity.** In this section, we study a variant of the normally hyperbolic case, in which the intersection of stable and unstable manifolds is no longer transverse, hence the results of [37] no longer apply. This is the case of a surface of rotation with a *degenerate hyperbolic* closed orbit.

Our example manifold is a topological torus $X = [-1, 1]_x \times \mathbb{S}^1_\theta$, equipped with the metric

$$ds^2 = dx^2 + A^2(x)d\theta^2 \tag{5.2}$$

where near $x = 0$,

$$A(x) = (1 + |x|^{2m})^{\frac{1}{2m}}$$

and m is an integer $\geqslant 1$. This manifold has a "fatter" part and a "thinner" part. At the thickest, there is an elliptic geodesic, and at the thinnest part, where $x = 0$, there is an unstable geodesic, which we denote by γ. If $m \geqslant 2$, the Gaussian curvature is chosen to vanish to a finite order at the unstable geodesic, hence the geodesic is *degenerately hyperbolic*. If the Gaussian curvature is strictly negative ($m = 1$) in a neighborhood of the thinnest part, the geodesic is non-degenerate; the geometry of a single closed hyperbolic geodesic has been extensively studied in [3, 4, 6, 7, 10, 13, 14] and others. As is seen in the previous section, in this non-degenerate hyperbolic case the energy decays sub-exponentially with derivative loss; in [3], it is shown that the sub-exponential decay rate is sharp. Based on the sharp polynomial loss in local smoothing and resolvent estimates in [12], Theorem 1.1 shows that for the degenerate hyperbolic periodic geodesic, we have the following estimates.

Theorem 5.3. *Let X be as above, and suppose $a(x)$ controls X geometrically outside a sufficiently small neighborhood $U \supset \gamma$, so that $\mathcal{N} = \{\gamma\}$. Then*

$$\|(h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}u\| \leqslant Ch^{-2m/(m+1)}\|u\|$$

for $z \in [1 - \delta, 1 + \delta] + i[-c_0, c_0]h^{2m/(m+1)}$.

As in the previous subsection, we deduce from this resolvent estimate an energy decay estimate for solutions to the damped wave equation.

Corollary 5.4. *Assume the hypotheses of Theorem 5.3 hold, and let u be a solution to (1.2) with $u_0 = 0$, and $u_1 \in H^s$ for some $s \in (0, 2]$. Then there exists a constant*

$C = C_s > 0$ *such that*

$$E(u,t) \leqslant C \left(\frac{t^{\frac{m+1}{m-1}}}{(\log(2+t))^{\frac{3(m+1)^2}{2(m-1)^2}}} \right)^{-s} \|u_1\|^2_{H^s}.$$

5.3. **Hyperbolic trapped set with small topological pressure.** In this section, we assume that the trapped set $\mathcal{N}$ has a hyperbolic structure, and that the topological pressure of half the unstable Jacobian on the trapped set is negative. Roughly, this means that the set $\mathcal{N}$ is rather thin, or filamentary: in dimension 2, this is for instance equivalent to require that $\mathcal{N}$ has Hausdorff dimension < 2. The simplest case to have in mind is a single, closed hyperbolic orbit. We then build on [25] to get resolvent estimates near the trapped set, which we extend to the global manifold with our different methods.

We briefly recall here the above dynamical notions. By definition, the hyperbolicity of $\mathcal{N} \subset S^*X$ means that for any $\rho \in \mathcal{N}$, the tangent space $T_\rho\mathcal{N}$ splits into *flow*, *stable* and *unstable* subspaces

$$T_\rho\mathcal{N} = \mathbb{R}\mathsf{H}_p \oplus E^s_\rho \oplus E^u_\rho \,.$$

If X is of dimension d, the spaces E^s_ρ and E^u_ρ are $d-1$ dimensional, and are preserved under the flow map:

$$\forall t \in \mathbb{R}, \quad d\, \mathrm{e}^{t\mathsf{H}_p}(E^s_\rho) = E^s_{\mathrm{e}^{t\mathsf{H}_p}(\rho)}, \quad d\, \mathrm{e}^{t\mathsf{H}_p}(E^u_\rho) = E^u_{\mathrm{e}^{t\mathsf{H}_p}(\rho)}.$$

Moreover, there exist $C, \lambda > 0$ such that

$$\begin{aligned} &i) && \|d\, \mathrm{e}^{t\mathsf{H}_p}(v)\| \leqslant C\, \mathrm{e}^{-\lambda t}\, \|v\|, \quad \text{for all } v \in E^s_\rho,\ t \geqslant 0 \\ &ii) && \|d\, \mathrm{e}^{-t\mathsf{H}_p}(v)\| \leqslant C\, \mathrm{e}^{-\lambda t}\, \|v\|, \quad \text{for all } v \in E^u_\rho,\ t \geqslant 0. \end{aligned} \tag{5.3}$$

One can show that there exist a metric on T^*X call the *adapted metric*, for which one can take $C = 1$ in the preceding equations.

The above properties allow us to define the unstable Jacobian. The adapted metric on T^*X induces a volume form Ω_ρ on any d dimensional subspace of $T(T^*_\rho X)$. Using Ω_ρ, we can define the unstable Jacobian at ρ for time t. Let us define the weak-stable and weak-unstable subspaces at ρ by

$$E^{s,0}_\rho = E^s_\rho \oplus \mathbb{R}\mathsf{H}_p\,, \quad E^{u,0}_\rho = E^u_\rho \oplus \mathbb{R}\mathsf{H}_p.$$

We set

$$J^u_t(\rho) = \det d\, \mathrm{e}^{-t\mathsf{H}_p}|_{E^{u,0}_{\mathrm{e}^{t\mathsf{H}_p}(\rho)}} = \frac{\Omega_\rho(d\, \mathrm{e}^{-t\mathsf{H}_p}\, v_1 \wedge \cdots \wedge d\, \mathrm{e}^{-t\mathsf{H}_p}\, v_d)}{\Omega_{\mathrm{e}^{t\mathsf{H}_p}(\rho)}(v_1 \wedge \cdots \wedge v_d)}\,, \qquad J^u(\rho) := J^u_1(\rho),$$

where $(v_1, \dots, v_d)$ can be any basis of $E^{u,0}_{\mathrm{e}^{t\mathsf{H}_p}(\rho)}$. While we do not necessarily have $J^u(\rho) < 1$, it is true that $J^u_t(\rho)$ decays exponentially as $t \to +\infty$.

We denote by $\mathrm{Pr}_\mathcal{N}$ the topological pressure functional on the closed, invariant set $\mathcal{N}$. We briefly recall a definition, see [36], [25] for more material. If f is a continuous function on $\mathcal{N}$, n an integer and $\epsilon > 0$, define

$$Z_{n,\epsilon}(f) = \sup_{\mathcal{S}} \left\{ \sum_{\rho \in \mathcal{S}} \exp \sum_{k=0}^{n-1} f \circ \mathrm{e}^{k\mathsf{H}_p}(\rho) \right\}$$

where the supremum is taken over all the (ϵ, n) separated subsets $\mathcal{S}$. The topological pressure of f on $\mathcal{N}$ is then

$$\mathrm{Pr}_{\mathcal{N}}(f) := \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log Z_{n,\epsilon}(f)\,.$$

Our main assumption here is that the topological pressure of $\frac{1}{2}\log J^u$ on $\mathcal{N}$ is negative, namely:

$$\mathrm{Pr}_{\mathcal{N}}(\frac{1}{2}\log J^u) < 0\,.$$

For some $\delta > 0$ small enough, this imply the following resolvent estimate:

$$\forall z \in [1-\delta, 1+\delta], \quad \|(h^2\Delta_g + \mathrm{i}\, a - z)^{-1}\| \leqslant C\frac{|\log h|}{h} \tag{5.4}$$

This estimate is already contained in [25], modulo two minor simplifications in our case: the manifold is compact, and infinity is replaced with the absorbing potential $\mathrm{i}\, a$, which control everything outside the trapped set – [15] shows explicitly that the estimate, with the more complicated geometry at infinity, of [25] implies the slightly simpler complex absorption result. Using Theorem 1.1, we immediately deduce the following result:

Theorem 5.5. *Let X be a compact manifold, and suppose that $a \geqslant 0$ controls X except on $\mathcal{N}$, which is assumed to be hyperbolic with the property that*

$$\mathrm{Pr}_{\mathcal{N}}(\frac{1}{2}\log J^u) < 0\,.$$

For $\delta > 0$ small enough, there is h_0 and $c_0 > 0$ such that for $h \leqslant h_0$ and $z \in [1-\delta, 1+\delta] + \mathrm{i}[-c_0, c_0]\frac{h}{|\log h|}$ we have

$$\|(h^2\Delta_g + \mathrm{i}\, ha - z)^{-1}\| \leqslant C\frac{|\log h|}{h}\,.$$

In particular, there is no spectrum near the real axis in a region of size $h/|\log h|$. As the resolvent estimate is the same order as that in Theorem 5.1, we deduce the same energy decay estimates as in Corollary 5.2. We remark that a similar result has recently been obtained by Rivière in [27], without the hypothesis (1.7).

6. From resolvent estimates to the damped wave equation and energy decay

In this section, we show how to move from a high energy resolvent estimate to an energy decay estimate for the damped wave equation, proving Corollaries 5.2 and 5.4. To estimate the energy decay for the damped wave equation, as usual we rewrite it as a first-order evolution problem : if $\boldsymbol{u} = (u, \partial_t u)$ one can write (1.2) as

$$\partial_t \boldsymbol{u} = \mathrm{i}\,\mathcal{B}\boldsymbol{u}, \quad \mathcal{B} = \begin{pmatrix} 0 & -\mathrm{i}\,\mathrm{Id} \\ \mathrm{i}\,\Delta_g & \mathrm{i}\, a \end{pmatrix}. \tag{6.1}$$

The evolution group $\mathrm{e}^{\mathrm{i}\, t\mathcal{B}}$ maps initial data $(u_0, u_1) \in H := H^1(X) \times H^0(X)$ to a solution $(u, \partial_t u)$ of (6.1) where u solves (1.2). For $s > 0$, define

$$\|u\|_s := \|u_0\|_{H^{1+s}(X)} + \|u_1\|_{H^s(X)}$$

It is not hard to see that if we can prove

$$\left\|\mathrm{e}^{\mathrm{i}\, t\mathcal{B}}(1 - \mathrm{i}\,\mathcal{B})^{-s}\right\|^2_{L^2(X)\to L^2(X)} \leqslant f(t) \tag{6.2}$$

then we can deduce a decay rate for the energy :

$$E(u,t) \leqslant f(t)\|u\|_s^2$$

It turns out that we can obtain bounds such as (6.2) if we have estimates on the high-frequency resolvent $(\lambda - \mathcal{B})^{-1}$, $|\lambda| \to \infty$.

To see this, we recall the following setup from [9]. Now suppose $(\lambda - \mathcal{B})^{-1}$ continues holomorphically to a neighborhood of the region

$$\Omega = \left\{ \lambda \in \mathbb{C} : |\operatorname{Im}\lambda| \leqslant \left\{ \begin{array}{ll} C_1, & |\operatorname{Re}\lambda| \leqslant C_2 \\ P(|\operatorname{Re}\lambda|), & |\operatorname{Re}\lambda| \geqslant C_2, \end{array} \right\} \right\},$$

where $P(|\operatorname{Re}\lambda|) > 0$ and is monotone decreasing (or constant) as $|\operatorname{Re}\lambda| \to \infty$, $P(C_2) = C_1$, and assume for simplicity that $\partial\Omega$ is smooth. Assume

$$\|(\lambda - \mathcal{B})^{-1}\|_{H\to H} \leqslant G(|\operatorname{Re}\lambda|) \tag{6.3}$$

for $\lambda \in \Omega$, where $G(|\operatorname{Re}\lambda|) = \mathcal{O}(|\operatorname{Re}\lambda|^N)$ for some $N \geqslant 0$.

Theorem 6.1. *(See [9, Theorem 3].) Suppose $\mathcal{B}$ satisfies all the assumptions above, and let $k \in \mathbb{N}$, $k > N+1$. Then for any $F(t) > 0$, monotone increasing, satisfying*

$$F(t)^{(k+1)/2} \leqslant \exp(tP(F(t))), \tag{6.4}$$

there is a constant $C > 0$ such that

$$\left\| \frac{e^{i\,t\mathcal{B}}}{(1 - i\,\mathcal{B})^k} \right\|_{H\to H} \leqslant CF(t)^{-k/2}. \tag{6.5}$$

In all cases considered in this paper, we have semiclassical resolvent estimates

$$\|(h^2\Delta_g + i\sqrt{z}ha - z)^{-1}\|_{L^2\to L^2} \leqslant \frac{\alpha(h)}{h}, \quad z \sim 1 + i\,h/\alpha(h),$$

If we rescale

$$\tau^2 = \frac{z}{h^2},$$

then our resolvent estimates become

$$\|(\Delta_g + i\,\tau a - \tau^2)^{-1}\|_{L^2\to L^2} \leqslant \frac{\alpha(|\tau|^{-1})}{|\tau|}.$$

for $\operatorname{Im}\tau \sim (\alpha(|\operatorname{Re}\tau|^{-1}))^{-1}$. By interpolation, this implies for $0 \leqslant j \leqslant 2$,

$$\|(\Delta_g + i\,\tau a - \tau^2)^{-1}\|_{H^s\to H^{s+j}} \leqslant |\tau|^{j-1}\alpha(|\tau|^{-1}).$$

Hence, with $\mathcal{B}$ as above and $H = H^1 \times H^0$, a simple calculation yields

$$\|(\lambda - \mathcal{B})^{-1}\|_{H\to H} \leqslant \alpha(|\lambda|^{-1}).$$

For Corollary 5.2, we take $\alpha(|\lambda|^{-1}) = \log(2 + |\lambda|)$. Then $k = 2$ suffices, $P(r) = \log^{-1}(r)$, and hence we may take

$$F(t) = e^{t^{1/2}/C}.$$

This recovers the endpoint estimate $s = 2$. To get the intermediate estimates for $s \in (0,2)$ we interpolate with the trivial estimate

$$E(u,t) \leqslant E(u,0).$$

For Corollary 5.4, we have $\alpha(|\lambda|^{-1}) = |\lambda|^{(m-1)/(m+1)}$, $N = (m-1)/(m+1) < 1$, so that $k = 2$, and $P(r) = r^{(1-m)/(m+1)}$. We try

$$F(t) = \frac{t^s}{\log^q(t)},$$

and insist

$$t^{3s/2} \log^{-3q/2}(t) \leqslant \exp(tt^{s(1-m)/(m+1)} \log^{q(m-1)/(m+1)}(t)),$$

which is satisfied if

$$s = \frac{m+1}{(m-1)}$$

and

$$q = \frac{3(m+1)^2}{2(m-1)^2}.$$

Again interpolating with the trivial estimate proves the Corollary.

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