GROWTH AND ZEROS OF THE ZETA FUNCTION FOR HYPERBOLIC RATIONAL MAPS

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Abstract. This paper describes new results on the growth and zeros of the Ruelle zeta function for the Julia set of a hyperbolic expanding rational map. It is shown that the zeta function is bounded by $\exp(C_K|s|^\delta)$ in strips $|\mathrm{Re}\, s| \leq K$, where δ is the dimension of the Julia set. This leads to bounds on the number of zeros in strips (interpreted as the Pollicott-Ruelle resonances of this dynamical system). An upper bound on the number of zeros in polynomial regions $\{|\mathrm{Re}\, s| \leq |\mathrm{Im}\, s|^\alpha\}$ is given, followed by weaker lower bound estimates in strips $\{\mathrm{Re}\, s > -C, |\mathrm{Im}\, s| \leq r\}$, and logarithmic neighbourhoods $\{|\mathrm{Re}\, s| \leq \rho \log |\mathrm{Im}\, s|\}$. Recent numerical work of Strain-Zworski suggests the upper bounds in strips are optimal.

1. Introduction

The motivation for the estimates described in this paper comes from scattering resonances. In the case where the underlying fractal set is the limit set of a convex co-compact Schottky group there is a correspondence between zeros of the zeta function and scattering resonances of the classically trapped set (see [7, 12, 16, 19] for details).

In the case of the Julia set, we are primarily interested in counting zeros of the zeta function Z, which may be interpreted as the Pollicott-Ruelle resonances of this dynamical system. The most interesting case is the number of zeros in regions $\mathrm{Re}\, s > -C,\ |\mathrm{Im}\, s| < r$, which we will show is bounded above by $Cr^{1+\delta}$, with δ the dimension of the Julia set. We prove a weak, sublinear lower bound on the number of zeros in this region, an honest linear lower bound in logarithmic neighbourhoods of the imaginary axis, and conjecture that our upper bound is actually sharp. We also obtain the upper bound $Cr^{1+2\alpha+\delta(1-\alpha)}$ for the number of zeros in more general regions $|\mathrm{Re}\, s| \leq |\mathrm{Im}\, s|^\alpha$ for $\alpha \in (0, 1)$.

Similar to [16, 7], we consider the dynamical system associated to a hyperbolic expanding rational map f when the Julia set $\mathcal{J}$ associated to this map is a totally disconnected set.

We think of $\mathcal{J}$ as a subset of the sphere $\widehat{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ naturally identified with $\widehat{\mathbb{R}}^2 = \mathbb{R}^2 \bigcup \{\infty\}$. Then $|f'(z)|$ can be thought of as a map $|f'(z)| : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$, analytic in a neighbourhood U of $\mathcal{J}$. If $[f'(z)]$ is the holomorphic extension of $|f'(z)|$ to a map $[f'(z)] : \widetilde{U} \subset \widehat{\mathbb{C}}^2 = \mathbb{C}^2 \bigcup \{\infty\} \to \widehat{\mathbb{C}}$, where $\widetilde{U} \subset \widehat{\mathbb{C}}^2$ is a neighbourhood of

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$\mathcal{J} \times \{0\}$, we can define the transfer operator:

$$\mathcal{L}(s)u(z) = \sum_{w \in f^{-1}(z)} [f'(w)]^{-s} u(w). \tag{1.1}$$

We will show in the course of this paper that $\mathcal{L}(s)$ is a trace class operator on an appropriately chosen class of functions, and the Ruelle zeta function Z will be defined as

$$Z(s) = \det (I - \mathcal{L}(s)) . \tag{1.2}$$

The zeta function defined in a similar context was first studied in the famous work of Ruelle [13]. We will prove the following bound of Z in terms of δ, the Hausdorff dimension of $\mathcal{J}$:

Theorem 1. *Suppose $Z(s)$ is the zeta function defined by (1.2) for the function f. Then for any C_0, there exists C_1 such that for $|\mathrm{Re}\, s| \leq C_0$ we have*

$$|Z(s)| \leq C_1 \exp(C_1 |s|^\delta), \quad \delta = \dim \mathcal{J} \tag{1.3}$$

where δ is the dimension of the Julia set of f.

In [16] the same result is given for the case when $f(z) = z^2 + c$ for c real, $c < -2$, in which case the Julia set is a real Cantor-type set. Numerical results in [16] suggest this theorem is sharp. Using Theorem 1 and a dynamical formula for $Z(s)$ from Proposition 5.1, we derive several estimates, both lower and upper bounds on the number of zeros in various regions. Based on numerical evidence from [16], it appears the lower bounds are not optimal, and in closing we give an example to demonstrate the subtlety of this question and some of the problems in approaching it.

2. Review of Julia sets

In this section we review a few classical results about the geometry of Julia sets that will be used later in the paper. The interested reader should consult [4, 5] for further details.

The Julia set $\mathcal{J}$ for a rational map can be defined to be the closure of the set of repelling periodic points, hence $\mathcal{J}$ is compact in the sphere. It is easy to see [3] that $\mathcal{J}$ is backward and forward invariant: $\mathcal{J} = f(\mathcal{J}) = f^{-1}(\mathcal{J})$, and in fact $f^p(\mathcal{J}) = \mathcal{J}$ for $p = 1, 2, \ldots$. We are interested in the case where $\mathcal{J}$ is disconnected (and hence totally disconnected). The hypothesis that $\mathcal{J}$ be totally disconnected is necessary in what follows, as in the proof of the essential Proposition 4.1. In the simple setup where $f(z) = z^2 + c$ for $c \notin \mathcal{M}$, where $\mathcal{M}$ is the Mandelbrot set, it would be interesting to determine the behaviour of the zeta function as $\mathrm{dist}\,(c, \mathcal{M}) \to 0$. The assumption that f be hyperbolic means there exists an $n \geq 1$ such that $\inf\{|(f^n)'(z)| : z \in \mathcal{J}\} > 1$. In other words, some iterate of f is expanding on the whole set. A sometimes useful fact (see [17]) is that a rational function is hyperbolic if and only if $\overline{\mathrm{PCV}(f)} \bigcap \mathcal{J} = \emptyset$, where $\mathrm{PCV}(f) = \bigcup_{n \geq 0} f^n(\mathrm{Crit}\, f)$ is the forward propagation of the set of critical points of f. If f is hyperbolic, one can replace f with an appropriate iterate and assume that f is strictly expanding near $\mathcal{J}$, since this does not change the geometry of $\mathcal{J}$. However, for simplicity, we assume throughout that f itself is expanding near $\mathcal{J}$.

The most important properties of $\mathcal{J}$ are the those making it a "cookie-cutter" set in the sense of [5]. Roughly speaking, this is to mean that a small neighbourhood intersected with $\mathcal{J}$ looks more or less like $\mathcal{J}$. This is made precise in the following proposition:

Proposition 2.1. *$\mathcal{J}$ is a cookie-cutter set, that is, there exist constants $c > 0$, $r_0 > 0$ such that for each $r < r_0$ and $z_0 \in \mathcal{J}$ there is a map $g : B(z_0, r) \to \widehat{\mathbb{C}}$ such that $g(B(z_0, r) \bigcap \mathcal{J}) \subset \mathcal{J}$ satisfying*

$$c^{-1} r^{-1} |z - w| \leq |g(z) - g(w)| \leq c r^{-1} |z - w| . \tag{2.1}$$

To prove this, we will need the Koebe distortion theorem (see [3] for a proof):

Lemma 2.2 (Koebe distortion theorem)**.** *If g is univalent (analytic and one-to-one) on the unit disk in $\mathbb{C}$ with $g(0) = 0$ and $g'(0) = 1$, then*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |g'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3} \tag{2.2}$$

and

$$\frac{|z|}{(1 + |z|)^2} \leq |g(z)| \leq \frac{|z|}{(1 - |z|)^2} \tag{2.3}$$

We can get Proposition 2.1 from this by a simple argument. Since $\mathrm{Crit}\, f \bigcap \mathcal{J} = \emptyset$, there is $R > 0$ so that for each $z_0 \in \mathcal{J}$, $f(z)$ is univalent on $B(z_0, R)$. This implies f^n is also univalent, since $(f^n)'(z) = f'(f^{n-1}(z)) \cdot f'(f^{n-2}(z)) \cdots f'(z)$. We want to modify the estimate 2.2 to apply to a function G univalent on a disk of radius $\delta > 0$, say, with $G'(0) = M \neq 0$. For $\zeta \in \{|\zeta| < 1\}$, define $g(\zeta) := G(\delta\zeta)/M$. Then g satisfies the hypotheses of the lemma, and we now have:

$$|M| \frac{1 - \frac{1}{\delta}|z|}{\left(1 + \frac{1}{\delta}|z|\right)^3} \leq |G'(z)| \leq |M| \frac{1 + \frac{1}{\delta}|z|}{\left(1 - \frac{1}{\delta}|z|\right)^4} \tag{2.4}$$

which, for $|z| < \delta/2$ yields

$$\frac{|M|}{c} \leq |G'(z)| \leq |M| c \tag{2.5}$$

for some constant c. The argument is finished by setting $r_0 = R/2$ and noting that taking an appropriate iterate $G = f^n$ maps a ball $B(z_0, r)$ of radius $r < r_0$ centered at $z_0 \in \mathcal{J}$ into a larger fixed ball $B(0, S)$, say, with the property that $G(B(z_0, r) \bigcap \mathcal{J}) = \mathcal{J}$. Thus there is a point $z_1 \in B(z_0, r)$ such that $S/(2r) \leq |G'(z_1)| \leq 2S/r$. By conjugating with an appropriate Möbius transformation, we can assume $z_0 = z_1$, so that $C^{-1} r^{-1} \leq |G'(z)| \leq C r^{-1}$ as claimed.

3. The transfer operator on L^2 spaces

It is more convenient to define the Ruelle transfer operator in terms of the inverse branches to f. Suppose f is an m to 1 function, and let $g_i(z)$ for $i = 1, 2, \ldots, m$ be the branches of f^{-1}. Now we interpret $\mathcal{J}$ as a subset of $\widehat{\mathbb{R}}^2$ instead of $\widehat{\mathbb{C}}$ and view $g_i : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$ real analytic and $|g_i'| : U \subset \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$ analytic in a neighbourhood U about $\mathcal{J}$. Then it is clear that both g_i and $|g_i'|$ extend holomorphically to functions $g_i : \widetilde{U} \subset \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$ and $[g_i'] : \widetilde{U} \subset \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}$ for $\widetilde{U} \subset \widehat{\mathbb{C}}^2$ a neighbourhood of $\mathcal{J} \times \{0\}$. The Ruelle transfer operator can then be defined as

$$\mathcal{L}(s) u(z) = \sum_{i=1}^{m} \left[g_i'(z)\right]^s u(g_i(z)). \tag{3.1}$$

We will show that with an appropriately chosen neighbourhood about $\mathcal{J} \subset \widehat{\mathbb{C}}^2$ and an appropriately chosen class of functions u, $\mathcal{L}$ is trace class. We begin, as in [16], with a review of characteristic values of a compact operator $A : H_1 \to H_2$, where H_j's are Hilbert spaces. Define

$$\|A\| = \nu_0(A) \geq \nu_1(A) \geq \cdots \geq \nu_l(A) \to 0$$

to be the eigenvalues of $(A^*A)^{\frac{1}{2}} : H_1 \to H_1$. The min-max principle shows that

$$\nu_l(A) = \min_{\substack{V \subset H_1 \\ \operatorname{codim} V = l}} \max_{\substack{v \in V \\ \|v\|_{H_1} = 1}} \|Av\|_{H_2} \tag{3.2}$$

Now suppose $\{x_j\}_{j=0}^{\infty}$ is an orthonormal basis of H_1, then

$$\nu_l(A) \leq \sum_{j=l}^{\infty} \|Ax_j\|_{H_2}. \tag{3.3}$$

To see this we use $V_l = \operatorname{span}\{x_j\}_{j=l}^{\infty}$ in (3.2): for $v \in V_l$ we have, by the Cauchy-Schwartz inequality, and the inequality $\|\cdot\|_{\ell_2} \leq \|\cdot\|_{\ell_1}$,

$$\|Av\|_{H_2}^2 = \left\| \sum_{j=l}^{\infty} \langle v, x_j \rangle_{H_1} Ax_j \right\|^2 \leq \|v\|_{H_1}^2 \left(\sum_{j=l}^{\infty} \|Ax_j\|_{H_2} \right)^2,$$

from which (3.2) gives (3.3).

We will also need the *Weyl inequality* (see [14]), which states that if $H_1 = H_2$ and $\lambda_j(A)$ are the eigenvalues of A,

$$|\lambda_0(A)| \geq |\lambda_1(A)| \geq \cdots \geq |\lambda_l(A)| \to 0, \tag{3.4}$$

then for any N,

$$\prod_{l=0}^{N} (1 + |\lambda_l(A)|) \leq \prod_{l=0}^{N} (1 + |\nu_l(A)|). \tag{3.5}$$

If A is trace class, i.e. if $\sum_l \nu_l(A) < \infty$, then the determinant

$$\det(I + A) := \prod_{l=0}^{\infty} (1 + \lambda_l(A)),$$

is well defined and

$$|\det(I + A)| \leq \prod_{l=0}^{\infty} (1 + \nu_l(A)). \tag{3.6}$$

We also need the following standard inequality about characteristic values (see [14])

$$\nu_{l_1+l_2+1}(A + B) \leq \nu_{l_1+1}(A) + \nu_{l_2+1}(B) \tag{3.7}$$

Finally, we finish with an obvious equality: suppose $A_j : H_{1j} \to H_{2j}$ and we form $\bigoplus_{j=1}^J A_j : \bigoplus_{j=1}^J H_{1j} \to \bigoplus_{j=1}^J H_{2j}$, then

$$\sum_{l=0}^{\infty} \nu_l \left(\bigoplus_{j=1}^{J} A_j \right) = \sum_{j=1}^{J} \sum_{l=0}^{\infty} \nu_l(A_j). \tag{3.8}$$

Now we want to define the Hilbert space we will be working with. For $D \subset \widehat{\mathbb{C}}^2$ open, let $H^2(D) := \{u \text{ holomorphic in } D : \int_D |u(z)|^2 dm(z) < \infty\}$. Let D_i be a finite union of balls covering $\mathcal{J}_i = g_i(\mathcal{J})$ for $i = 1, 2, \ldots, m$ so that $D_i \cap D_j = \emptyset$ for $i \neq j$, and $D = \cup D_i$. We have the following proposition:

Proposition 3.1. *Suppose that $\mathcal{L}(s) : H^2(D) \to H^2(D)$ is defined by (3.1), with g_i the m inverse branches of f hyperbolic expanding rational. Then for all $s \in \mathbb{C}$, $\mathcal{L}(s)$ is trace class and*

$$|\det(I - \mathcal{L}(s))| \leq C \exp(C|s|^3) \tag{3.9}$$

for some constant C.

Proof. We write $H^2(D) = \bigoplus_{j=1}^m H^2(D_j)$ and $\mathcal{L}(s) = \bigoplus_{i,j=1}^m \mathcal{L}_{ij}(s)$, where

$$\mathcal{L}_{ij}(s)u(z) := [g_i'(z)]^s\, u(g_i(z)), \;\; z \in D_j. \tag{3.10}$$

Note that from (3.7) and (3.8) we certainly have

$$\nu_k(\mathcal{L}(s)) \leq m^2 \max_{1 \leq i,j \leq m} \nu_{[k/2m]}(\mathcal{L}_{ij}(s)).$$

Let $r_0 > 0$ be the minimum radius for which $|Dg_i(z)| < 1$, $i = 1, 2, \ldots, m$, on a ball of radius r_0 centered at a point of $\mathcal{J}$. Let

$$U = \bigcup_{i=1}^m U_i := \bigcup_{i=1}^m \{\mathcal{J}_i + B_{\widehat{\mathbb{C}}^2}(0, r)\}$$

for $r < r_0/2$. Let $M = \max_{\overline{U}} |Dg_i(z)| < 1$, and pick for D a finite cover of $\mathcal{J}$, $D = \bigcup_{i=1}^m D_i$ made up of balls of radius r centered at points of $\mathcal{J}$ as above, so that for each $z \in \mathcal{J}$, $d_{\widehat{\mathbb{C}}^2}(z, \partial D) \geq \frac{1+M}{2} r$, and D_i covers $\mathcal{J}_i = g_i(\mathcal{J})$. Then for any point $z \in D_i$, $|z - w| < r$ for some $w \in \mathcal{J}$, so $|g_j(z) - g_j(w)| \leq Mr$, so that $d_{\widehat{\mathbb{C}}^2}(g_j(D_i), \partial D_j) \geq \left(\frac{1+M}{2} - M\right) r > 0$. Lemmas 3.2 and 3.3 together with the estimate $|[g_i'(z)]^s| \leq e^{C|s|}$ now give for some C_1:

$$\nu_l(\mathcal{L}_{ij}(s)) \leq C_1 e^{C|s| - l^{1/2}/C_1}.$$

With this in hand, we see that (3.6) implies

$$\det(I - \mathcal{L}(s)) \leq \prod_{l=0}^{\infty} \left(1 + Ce^{C|s| - l^{1/2}/C}\right) \leq Ce^{C^5|s|^3}$$

so that $\mathcal{L}(s)$ is trace class as claimed. To finish with the proposition, we need the following two lemmas, taken almost directly from [7]. □

Lemma 3.2. *Let $\rho \in (0,1)$ and $R^\rho : H^2(B_{\mathbb{C}^2}(0,1)) \to H^2(B_{\mathbb{C}^2}(0,\rho))$ induced by the restriction map of $B_{\mathbb{C}^2}(0,1)$ to $B_{\mathbb{C}^2}(0,\rho)$. Then for any $\widetilde{\rho} \in (\rho, 1)$ there exists a constant C such that*

$$\nu_l(R^\rho) \leq C \widetilde{\rho}^{\,l^{1/2}}.$$

Proof. Using (3.3) with the standard basis $(x_\alpha)_{\alpha \in \mathbb{N}^2}$ for $H^2(B_{\mathbb{C}^2}(0,1))$ given by

$$x_\alpha(z) = c_\alpha z_1^{\alpha_1} z_2^{\alpha_2}, \quad \int_{B_{\mathbb{C}^2}(0,1)} |x_\alpha(z)|^2 dz = 1, \;\; \alpha \in \mathbb{N}^2 \tag{3.11}$$

for which we have

$$\|R^\rho(x_\alpha)\|^2 = \int_{B_{\mathbb{C}^2}(0,\rho)} |x_\alpha(z)|^2 dz = \rho^{2|\alpha|+4}.$$

The number of α's for which $|\alpha| \leq m$ is bounded by $(m+1)^2$, so by (3.3)

$$\nu_l(R^\rho) \leq \sum_{|\alpha|\geq l} \rho^{|\alpha|+2} \leq C \sum_{k \geq l^{1/2}} (k+1)^2 \rho^k \leq C \widetilde{\rho}^{\,l^{1/2}}.$$

□

Lemma 3.3. *Suppose $\Omega_j \subset \mathbb{C}^2$, $j = 1, 2$ are open sets and $\Omega_1 = \bigcup_{k=1}^K B_{\mathbb{C}^2}(z_k, r_k)$. Let g be a holomorphic mapping defined on a neighbourhood $\widetilde{\Omega}_1$ of Ω_1 taking values in Ω_2 satisfying*

$$d_{\mathbb{C}^2}(g(\Omega_1), \partial\Omega_2) > \frac{1}{C_0} > 0, \quad 0 < \|Dg(z)\| < 1, \ z \in \widetilde{\Omega}_1.$$

If

$$A : H^2(\Omega_2) \to H^2(\Omega_1), \quad Au(z) := u(g(z)), \ z \in \Omega_1$$

then for some C_1 depending only on K, $d_{\mathbb{C}^2}(g(\Omega_1), \partial\Omega_2)$, $\sup_{\widetilde{\Omega}_1} \|Dg(z)\|$, we have

$$\nu_l(A) \leq C_1 e^{-l^{1/2}/C_1}$$

where $\nu_l(A)$'s are the characteristic values of A.

Proof. Define a new Hilbert space

$$\mathcal{H} := \bigoplus_{k=1}^K H^2(B_k), \quad B_k = B_{\mathbb{C}^2}(z_k, r_k),$$

and a natural operator

$$J : H^2(\Omega_1) \to \mathcal{H}, \quad (Ju)_k = u|_{B_k}$$

We claim $J^*J : H^2(\Omega_1) \to H^2(\Omega_1)$ is invertible, with constants depending only on K. To see this, note that for any $u \in H^2(\Omega_1)$,

$$\|u\|^2 \leq \langle Ju, Ju \rangle_{\mathcal{H}} \leq K \|u\|^2 .$$

Hence

$$\|J^*Ju\|^2 = \langle Ju, JJ^*Ju \rangle_{\mathcal{H}} \leq K \|u\| \|J^*Ju\| \tag{3.12}$$

and

$$\|u\|^2 \leq \langle J^*Ju, u \rangle_{H^2(\Omega_1)} \leq \|J^*Ju\| \|u\| \tag{3.13}$$

for any $u \in H^2(\Omega_1)$. The estimate (3.12) implies J^*J is bounded, while the estimate (3.13) implies J^*J is one-to-one. Since any one-to-one self-adjoint operator is also onto, J^*J is invertible, and furthermore,

$$\frac{1}{K} \|u\|^2 \leq \left\|(J^*J)^{-1}u\right\| \leq \|u\|^2.$$

Thus we calculate,

$$\nu_l(A) = \nu_l((J^*J)^{-1}J^*JA) \leq \left\|(J^*J)^{-1}\right\| \|J^*\| \nu_l(JA).$$

Note then that

$$\nu_l(JA) \leq K \max_{1 \leq k \leq K} \nu_{[l/K]}(A_k),$$

where

$$A_k : H^2(\Omega_2) \to H^2(B_k), \;\; A_k u(z) = u(g_k(z)), \;\; g_k = g|_{B_k}$$

In order to estimate the characteristic values for A_k, note we can extend g_k to a larger ball in $\widetilde{\Omega}_1$, $\widetilde{B}_k$ such that the image of its closure is still in Ω_2. That gives us the operators $R_k : H^2(\widetilde{B}_k) \to H^2(B_k)$, $R_k u = u|_{B_k}$, and $\widetilde{A}_k$ defined similarly to A_k with B_k's replaced with $\widetilde{B}_k$'s. Now we have $A_k = R_k \widetilde{A}_k$ which implies

$$\nu_l(A_k) \leq \left\| \widetilde{A}_k \right\| \nu_l(R_k).$$

Lemma 3.2 gives $\nu_l(R_k) \leq C_2 e^{-l^{1/2}/C}$. To see these constants don't depend on the r_k's, note that the proof of Lemma 3.2 scales to the case of $R^\rho : B(0, r_k) \to B(0, \rho r_k)$ without modifying the constants, and this completes the proof. $\square$

4. Estimates in terms of the dimension of $\mathcal{J}$

In order to prove Theorem 1 we need a few more important facts. Recall that the diameter of a set E is defined as $\operatorname{diam}(E) = \sup\{|x-y| : x, y \in E\}$.

Proposition 4.1. *Let $\mathcal{J} \in \widehat{\mathbb{C}}$ be the Julia set for f hyperbolic expanding rational, and assume $\mathcal{J}$ is totally disconnected. Then there exist constants $K = K(c)$ and δ_0 such that for $\delta < \delta_0$ the connected components of $\mathcal{J} + \overline{B}_{\widehat{\mathbb{C}}}(0, \delta)$ have diameter at most $K\delta$.*

Proof. Let c and r_0 be as in Proposition 2.1. Since $\mathcal{J}$ is totally disconnected, there exists $\epsilon_0 > 0$ such that $\widehat{\mathcal{J}} = \mathcal{J} + B(0, \epsilon_0)$ has more than one connected component, and every connected component of $\widehat{\mathcal{J}}$ has diameter at most $(4c)^{-1}$. Then we apply Proposition 2.1 with $r = c\delta\epsilon_0^{-1}$, with $\delta \leq \delta_0 < r_0 \epsilon_0 c^{-1}$. The function g guaranteed by Proposition 2.1 takes points in $\mathcal{J}$ to points in $\mathcal{J}$, so if $z \in \mathcal{J}$, $g(B(z, \delta)) \subset B(g(z), \epsilon_0)$. Thus a connected component of $\mathcal{J} + B(0, \delta)$ is mapped into a connected component of $\widehat{\mathcal{J}}$. Now suppose $d(z, w) > r/2$. Then

$$d(g(z) - g(w)) \geq \frac{1}{cr} d(z, w) \geq \frac{1}{2c} > \frac{1}{4c}$$

so that $g(z)$ is in a different connected component from $g(w)$. Hence z and w must have been in separate connected components, and we conclude the diameter of the connected component containing z is at most $K\delta = r$. $\square$

We have a bound on the diameter of the connected components of $\mathcal{J} + \overline{B}_{\widehat{\mathbb{C}}}(0, \delta)$, but eventually we will need to cover $\mathcal{J}$ by balls, uniformly finite in δ so that we may again apply Lemma 3.3.

Lemma 4.2. *Suppose $D \subset \widehat{\mathbb{C}}$ is a compact set with the property that all connected components of $E = D + B_{\widehat{\mathbb{C}}}(0, \delta)$ have diameter bounded by $K\delta$. Then for any $\lambda \in (0, 1)$ and any connected component E_i of E, there exists a cover $U_i = U_i(\delta) \subset E_i$ of $D_i = E_i \bigcap D$ by at most $K' = K'(\lambda)$ balls of radius δ centered at points of D_i such that $d_{\widehat{\mathbb{C}}}(z, \partial U_i) \geq \lambda\delta$ for $z \in D_i$.*

Proof. Let $l = (1-\lambda)/\sqrt{2}$. If E_i is a connected component of E, then it fits in a closed ball of diameter $K\delta$ by hypothesis. A ball of diameter $K\delta$ is contained in a closed cube Q of side length $K\delta$, which can be covered by $K'(\lambda)$ closed cubes of side length $l\delta$ by simply starting at one corner of Q and covering it with cubes $\{q_k\}$ of side length $l\delta$ intersecting only on their boundaries. For each k, if $D \bigcap q_k \neq \emptyset$, select any point $p_k \in D \bigcap q_k$; if the intersection is empty, select nothing. Then set $U_i = \bigcup_{k=1}^{K'} B(p_k, \delta)$. A simple calculation gives for any $z \in D_i$, $z \in q_k$ for some k, giving

$$d_{\widehat{\mathbb{C}}}(z, \partial U_i) \geq d_{\widehat{\mathbb{C}}}(p_k, \partial U_i) - d_{\widehat{\mathbb{C}}}(p_k, z) \geq \lambda \delta$$

since $z \in q_k$ implies, in particular, that $d_{\widehat{\mathbb{C}}}(p_k, z) \leq \sqrt{2} l \delta$. □

Remark. It is clear that Lemma 4.2 extends to $\widehat{\mathbb{C}}^n$ with constants depending only on K and the dimension.

Proof of Theorem 1. As in [16] choose $h = |s|^{-1}$ where $|s|$ is large, but $|\mathrm{Re}\,(s)|$ is bounded. Now viewing $\mathcal{J}$ as a subset of $\widehat{\mathbb{R}}^2$ instead of $\widehat{\mathbb{C}}$, form $\widetilde{\mathcal{J}}(h) = \mathcal{J} + B_{\widehat{\mathbb{C}}^2}(0, h)$. Proposition 4.1 tells us the diameter of each connected component of $\widetilde{\mathcal{J}}(h)$ has diameter less than Kh. Since g_i (now thought of as the holomorphic extension $g_i : \widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$) is a contraction near $\mathcal{J}$, there is some h_0 so that if $h < h_0$, $M = \max |Dg_i(z)| < 1$ for z in the closure of $\mathcal{J} + B_{\widehat{\mathbb{C}}^2}(0, h)$. Let $\beta = \frac{1}{2}(M + 1)$, and suppose there are $P(h)$ connected components of $\widetilde{\mathcal{J}}(h)$. Using Lemma 4.2 we can pick a subcover $U(h) = \bigcup_{i=1}^m U_i(h)$ of at most $K'P(h)$ balls contained in $\widetilde{\mathcal{J}}$ and centered at points of $\mathcal{J}$ satisfying

$$d_{\widehat{\mathbb{C}}^2}(z, \partial U) \geq \beta h, \quad z \in \mathcal{J}.$$

Since any point of U is within h of some $z \in \mathcal{J}$ and we know $g_i : \mathcal{J}_j \to \mathcal{J}_i$,

$$d_{\widehat{\mathbb{C}}^2}(g_i(U_j(h)), \partial U_i(h)) \geq (\beta - M)h > C^{-1}h$$

for some constant C independent of h.

It is classical that the Hausdorff measure of the Julia set is finite (see [17] and the references therein) and that the Hausdorff dimension equals the box-counting dimension. Using the setup of the box-counting dimension, let $N(\epsilon)$ be the number of sets of diameter ϵ needed to cover $\mathcal{J}$. With

$$\delta = - \lim_{\epsilon \to 0^+} \frac{\log N(\epsilon)}{\log \epsilon},$$

$P(h) = N(Kh)$ implies $P(h) = \mathcal{O}(h^{-\delta})$. We write $\mathcal{L}(s)$ as a sum of m^2 operators $\mathcal{L}_{ij}(s)$ as before, $\mathcal{L}_{ij}(s) : H^2(U_i) \to H^2(U_j)$, but now we have a better bound on the weight independent of h. Recall $[g_i'(z)] : \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}$, and we are only interested in values of $[g_i'(z)]$ on $U_j(h)$, so $|\arg[g_i'(z)]| \leq |\mathrm{Im}\, z| \leq h = |s|^{-1}$. Hence,

$$\begin{aligned} |[g_i'(z)]^s| &\leq C \exp(|s| |\arg [g_i'(z)]|) \\ &\leq \exp\left(C_1 |s| \left(|\mathrm{Im}\,(z_1)|^2 + |\mathrm{Im}\,(z_2)|^2 \right)^{1/2} \right) \\ &\leq C_2, \quad z \in U_j(h). \end{aligned} \tag{4.1}$$

Each $\mathcal{L}_{ij}(s)$ is a sum of no more than $P(h)$ operators, each of which satisfies $\nu_l \leq C\alpha^{l^{1/2}/C}$ for some $0 < \alpha < 1$ by Lemma 3.3. Thus using again (3.7) and (3.8) we get the estimate

$$\log |\det (I - \mathcal{L}(s))| \leq CP(h) = \mathcal{O}(h^{-\delta}) \tag{4.2}$$

which is (1.3). □

5. Counting Zeros in Strips

In this section we prove the following corollary to Theorem 1. The methods used here are similar to those used in [16] and [10].

Corollary 2. *Let $m(s)$ denote the multiplicity of a zero of $Z(s)$ at s. Then*

$$\sum \{m(s) : r \leq |\mathrm{Im}\, s| \leq r+1, \quad \mathrm{Re}\, s > -C_0\} \leq C_1 r^{\delta} \tag{5.1}$$

where $\delta = \dim \mathcal{J}$.

In order to prove this corollary, we will need to bound $Z(s)$ away from zero for $\mathrm{Re}\, s \geq C_0$. We do this by employing a dynamical formula for $Z(s)$ which is interesting in its own right. For the development of this dynamical formula, we take D_i to be $\widehat{\mathbb{C}}^2$-balls containing $\mathcal{J} \subset \widehat{\mathbb{R}}^2$. We again view f as a map $f : \widehat{\mathbb{R}}^2 \to \widehat{\mathbb{R}}^2$ and then extend to a holomorphic function $\widehat{\mathbb{C}}^2 \to \widehat{\mathbb{C}}^2$ and write f for this extension whenever unambiguous.

Proposition 5.1. *For* $\mathrm{Re}\,(s) \gg 0$,

$$\det(I - \mathcal{L}(s)) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} \sum_{f^n(z)=z} \frac{[(f^n)'(z)]^{-s}}{\left|\det(I - (d\ (f^n)\,(z))^{-1})\right|} \right). \tag{5.2}$$

Proof. For $|\lambda|$ sufficiently small, $\log(I - \lambda\mathcal{L}(s))$ is well defined and

$$\det(I - \lambda\mathcal{L}(s)) = \exp\left(-\sum_{n=1}^{\infty} \frac{\lambda^n}{n} \mathrm{tr}\,(\mathcal{L}(s)^n) \right).$$

In order to evaluate the traces, we write

$$\mathrm{tr}\,\mathcal{L}(s)^n = \sum_{(i_1,\dots,i_{n+1})} \mathrm{tr}\ \left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_{n+1}}(s)\right),$$

where $\mathcal{L}_{ij}(s)$ is given by (3.10). If the target space is different from the domain space, there are no eigenvalues, so that

$$\sum_{(i_1,\dots,i_{n+1})} \mathrm{tr}\ \left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_{n+1}}(s)\right) = \sum_{(i_1,\dots,i_n)} \mathrm{tr}\ \left(\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_1}(s)\right).$$

We have

$$\mathcal{L}_{i_1 i_2}(s) \circ \cdots \circ \mathcal{L}_{i_n i_1}(s) u(z) = [(g_{i_1} \circ \cdots \circ g_{i_n})'(z)]^s\, u(g_{i_1} \circ \cdots \circ g_{i_n}(z)),$$

and Lemma A.1 in the appendix shows that

$$\mathrm{tr}\ ((g_{i_1} \circ \cdots \circ g_{i_n})^*) = \frac{1}{|\det(I - d\,(g_{i_1} \circ \cdots \circ g_{i_n})(z))|}$$

which completes the proof once we put $\lambda = 1$. □

Proof of Corollary 2. Using (5.2) it is clear that for $\mathrm{Re}\, s \geq C_0$ we have

$$|[(f^n)'(z)]^{-s}| \leq C C_1^{-n\mathrm{Re}\,(s)}$$

with $C_1 > 1$ since z is a periodic repeller and $[(f^n)'(z)]$ is real on $\mathcal{J}$. Then the convergence of the double series in (5.2) is immediate and gives for $\mathrm{Re}\, s \geq C_0$,

$Z(s) \geq 1/2$. With Z zero free for $\operatorname{Re} s \geq C_0$, an application of the Jensen formula shows the left hand side of (5.1) is bounded by

$$\begin{aligned} \sum \{m(s) : |s - ir - C_0| \leq 2(C_2 + C_0)\} &\leq C_3 \max_{|s-ir-C_0|\leq 4C_2} \log |Z(s)| \\ &\leq C \max_{\substack{|\operatorname{Re} s|\leq C_3 \\ |s|\leq 4C_2+r}} \log |Z(s)| \\ &\leq C_1 r^{\delta}. \end{aligned}$$

□

6. Polynomial Neighbourhoods

Suppose $\{\mu_j\}$ are the zeros of $Z(s)$ counted with multiplicity. Let $0 < \alpha < 1$ and consider the region $R_\alpha = \{|\operatorname{Re} s| \leq |\operatorname{Im} s|^\alpha, |s| \geq 1\}$. Let $N_\alpha(r) = \#\{\mu_j \in R_\alpha : |\mu_j| \leq r\}$. We expect $N_\alpha(r)$ to be somewhere in between the upper bound in strips, $N_0(r) = \#\{\mu_j : \operatorname{Re} \mu_j > C_0,\ |\operatorname{Im} \mu_j| \leq r\} \leq Cr^{1+\delta}$, and the global bound $N_1(r) = \#\{|\mu_j| \leq r\} \leq Cr^3$. The following theorem asserts we get the expected interpolation. The techniques used in the proof of theorem should extend easily to the case of convex co-compact Schottky groups [7], giving the same upper bound as in [18].

Theorem 3. *With $\{\mu_j\}$, α, and N_α defined as above,*

$$N_\alpha(r) \leq C_\alpha r^{1+2\alpha+\delta(1-\alpha)} \tag{6.1}$$

Proof. In order to begin, we start, as in the proof of Theorem 1 by constructing a cover of $\mathcal{J}$ by open sets. For $h = |s|^{-1}$, let $B_\alpha(h)$ be the open ball

$$B_\alpha(h) := B_{\mathbb{C}^2}(0, h^{1-\alpha}),$$

and set $\widetilde{\mathcal{J}}(h) = \mathcal{J} + B_\alpha(h)$. We can pick a finite subcover of $\widetilde{\mathcal{J}}(h)$, $U(h) = \bigcup_1^m U_i(h)$, as before, so that

$$d(g_i(U_j(h), \partial U_i(h)) \geq C_\alpha^{-1} h^{1-\alpha}$$

for some constant C_α independent of h. Write $\mathcal{L}$ as a sum of m^2 operators $\mathcal{L}_{ij}(s) : H^2(U_i) \to H^2(U_j)$ as before. Since $[g_i'(z)] : \widehat{\mathbb{R}}^2 \to \mathbb{R}$, if we take $s \in R_\alpha' := \{|\operatorname{Re} s| \leq 5|\operatorname{Im} s|^\alpha\}$,

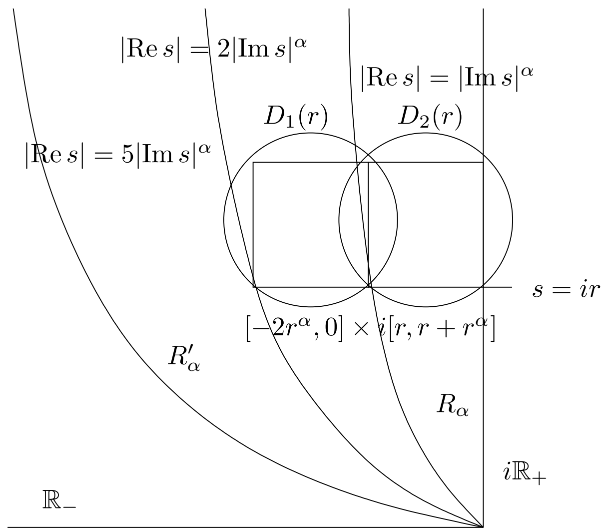
$$\begin{aligned} |[g_i'(z)]^s| &\leq C \exp(C|\operatorname{Re} s| |\log(|[g_i'(z)]|) + C|\operatorname{Im} s| |\arg[g_i'(z)]|) \\ &\leq C \exp\left(C|s|^\alpha + C|\operatorname{Im} s| \left((\operatorname{Im} z_1)^2 + (\operatorname{Im} z_2)^2\right)^{\frac{1}{2}}\right) \\ &\leq C \exp(C|s|^\alpha). \end{aligned} \tag{6.2}$$

Now each $\mathcal{L}_{ij}(s)$ is a sum of no more than $P(h) = \mathcal{O}(h^{-\delta(1-\alpha)})$ operators, each of which has characteristic values $\{\nu_l\}$ satisfying $\nu_l \leq Ce^{C|s|^\alpha - l^{1/2}/C}$. Then for $s \in R_\alpha'$,

$$\begin{aligned} |Z(s)| &\leq \prod \left(1 + Ce^{C|s|^\alpha - l^{1/2}/C}\right)^{|s|^{\delta(1-\alpha)}} \\ &\leq Ce^{C^5|s|^{3\alpha+\delta(1-\alpha)}}. \end{aligned}$$

Next we restrict attention to zeros in the upper left quadrant. Observe that for any $r_1 < r_2$, we can cover $R_\alpha \bigcap \{r_1 \leq |s| \leq r_2\}$ by boxes of width $2r^\alpha$ and height r^α with right bottom corner at $s = ir$, for $r_1 < r < r_2$. If $n_B(r)$ is the number

Figure 1. Regions used in the proof of Theorem 3



of such boxes needed to cover R_α, clearly $r^{-\alpha}/2 \leq dn_B(r) \leq 2r^{-\alpha}$. Each box $[-2r^\alpha, 0] \times i[r, r + r^\alpha]$ can be covered by two discs $D_1(r) \bigcup D_2(r)$ with

$$\begin{aligned} D_1(r) &= D(-\frac{3}{2}r^\alpha + i(r + \frac{1}{2}r^\alpha), \frac{1}{\sqrt{2}}r^\alpha), \\ D_2(r) &= D(-\frac{1}{2}r^\alpha + i(r + \frac{1}{2}r^\alpha), \frac{1}{\sqrt{2}}r^\alpha), \end{aligned}$$

both of which fit inside of R'_α (see Figure 1). The Jensen formula tells us $N_D(r) := \#\{\mu_j : \mu_j \in D_1(r) \bigcup D_2(r)\} \leq Cr^{3\alpha+\delta(1-\alpha)}$. Then

$$N_\alpha(r) \leq \int_1^r N_D(s) dn_B(s) \leq Cr^{1+2\alpha+\delta(1-\alpha)}$$

as claimed. □

7. Lower Bounds on the Number of Zeros

In order to prove lower bounds on the number of zeros, we will use extensively the dynamical formula (5.2). In light of [11] we see the series in Proposition 5.1 actually converges for all $\mathrm{Re}\, s > \delta$. We will use this in the following proofs when we select our contours of integration. Let $w(s) = Z(is + \delta)$, and suppose $\{\lambda_j\}_{j=1}^\infty$ are the zeros of w counted with multiplicity. Let $u_1(t) \in \mathcal{D}'(\mathbb{R}_+)$ be the distribution

$$u_1(t) := \sum_j e^{it\lambda_j} \tag{7.1}$$

and let $u_2(t) \in \mathcal{D}'(\mathbb{R}_+)$ be the distribution

$$u_2(t) := te^{-\delta t} \sum_n \frac{1}{n} \sum_{f^n(z)=z} \frac{\delta_0(t - L_n(z))}{|\det(I - d\,(f^n)^{-1}(z))|} \tag{7.2}$$

where δ_0 is the usual Dirac mass and

$$L_n(z) = \log[(f^n)'(z)].$$

Lemma 7.1. *With u_1, u_2 as above,*

$$u_1(t) = u_2(t) \tag{7.3}$$

in the sense of distributions on $\mathbb{R}_+$.

Remark. We use this distribution identity to make the presentation of the following proofs clear, and in order to quote directly Lemma 7.2 below from [15].

Proof. Let $w_\epsilon(s) = Z(is + \delta + \epsilon)$ for $\epsilon > 0$. Then w_ϵ has a dynamical expansion for $\operatorname{Im} s < 0$, and if $\{\lambda_j^\epsilon\}$ are the zeros of w_ϵ counted with multiplicity, then $\operatorname{Im} \lambda_j^\epsilon > 0$ for each j. In light of Proposition 3.1, we see $w(s)$ is an entire function of order 3, hence the Weierstrass factorization gives

$$\frac{d^3}{d\lambda^3}\left(\frac{w'(\lambda)}{w(\lambda)}\right) = -3! \sum_j \frac{1}{(\lambda_j - \lambda)^4}. \tag{7.4}$$

Now

$$-3! \frac{1}{(\lambda_j - \lambda)^4} = i \frac{d^3}{d\lambda^3} \int_0^\infty e^{it(\lambda_j - \lambda)} dt = -\int_0^\infty t^3 e^{it(\lambda_j - \lambda)} dt$$

provided $\operatorname{Im} \lambda_j > 0$ and λ is real. Hence the right hand side of (7.4) is

$$v_1^\epsilon(t) = -\mathcal{F}\left(t^3 \sum_j (e^{it\lambda_j^\epsilon})_+\right) \tag{7.5}$$

and the right hand side of (7.4) is

$$v_2^\epsilon(t) = -\mathcal{F}\left(t^4 e^{-(\delta+\epsilon)t} \sum_n \frac{1}{n} \sum_{f^n(z)=z} \frac{\delta_0(t - L_n(z))}{|\det(I - d\,(f^n)^{-1}(z))|}\right), \tag{7.6}$$

where $\mathcal{F}$ denotes the usual Fourier transform. Since both of these distributions are tempered and $\mathcal{F}$ is an isomorphism on $\mathcal{S}'$, we conclude

$$u_1^\epsilon(t) = u_2^\epsilon(t)$$

away from 0, where

$$u_1^\epsilon(t) := \sum_j e^{it\lambda_j^\epsilon} \tag{7.7}$$

and

$$u_2^\epsilon(t) := te^{-(\delta+\epsilon)t} \sum_n \frac{1}{n} \sum_{f^n(z)=z} \frac{\delta_0(t - L_n(z)}{|\det(I - d\,(f^n)^{-1}(z))|}. \tag{7.8}$$

Finally, integrating against a test function in $C_0^\infty(\mathbb{R}_+)$ and sending ϵ to zero gives (7.3). $\square$

Next we use this distribution identity to count zeros in specific regions.

7.1. **Zeros in Strips.** We first recall some notation: we say $f(x) = \Omega(g(x))$ as $x \to \infty$ if there does not exist any constant C for which $f(x) \le Cg(x)$ as $x \to \infty$. That is, $f(x)$ cannot be controlled by $g(x)$ as $x \to \infty$.

Theorem 4. *Let δ be the dimension of the Julia set, $Z(s)$ the dynamical zeta function, and $\{\mu_j\}$ the zeros of Z with multiplicity. Then there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, $\#\{\mu_j : |\mathrm{Im}\,\mu_j| \le r,\ \ -C\epsilon^{-1} < \mathrm{Re}\,\mu_j < \delta\} = \Omega(r^{1-\epsilon})$.*

Observe Corollary 2 implies an upper bound on the number of zeros in strips:

$$\#\left\{\mu_j : |\mathrm{Im}\,\mu_j| \le r, -C < \mathrm{Re}\,\mu_j < \delta\right\} \le Cr^{1+\delta}$$

which suggests this lower bound is in fact not optimal. Instead we have the following conjecture.

Conjecture 5. *There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there exists $0 < C_\epsilon < \infty$ such that*

$$\#\{\mu_j : |\mathrm{Im}\,\mu_j| \le r,\ \ -\epsilon^{-1} < \mathrm{Re}\,\mu_j < \delta\} \ge (C_\epsilon^{-1})r^{1+\delta},$$

where δ is the dimension of the Julia set.

Proof of Theorem 4. Let φ satisfy $\widehat{\varphi} \in C_0^\infty(\mathbb{R})$, $\widehat{\varphi}(0) = 1$, $\widehat{\varphi} \ge 0$ and $\mathrm{supp}\,\widehat{\varphi} \subset [-1, 1]$, where $\widehat{\varphi}$ denotes the Fourier transform. Define $\widehat{\varphi}_{\gamma,d}(t) = \widehat{\varphi}(\gamma^{-1}(t-d)$ with $d \ge 1$, $\gamma \le 1$, so that $\mathrm{supp}\,\widehat{\varphi}_{\gamma,d} \subset [d-\gamma, d+\gamma] \subset \mathbb{R}_+$. Then

$$\langle u_1, \widehat{\varphi}_{\gamma,d}\rangle = (2\pi)^{-1/2}\sum_j \varphi_{\gamma,d}(\lambda_j) \tag{7.9}$$

and

$$\langle u_2, \widehat{\varphi}_{\gamma,d}\rangle = \sum_n \frac{1}{n}\sum_{f^n(z)=z} \frac{L_n(z)\exp(-L_n(z)\delta)}{|\det(I - d\,(f^n)^{-1}(z))|}\widehat{\varphi}_{\gamma,d}(L_n(z)) \tag{7.10}$$

with $L_n(z) = \log[(f^n)'(z)]$. If d is chosen near one of the $L_n(z)$s and γ is small, (7.10) is bounded from below by

$$C^{-1}de^{-\delta d}. \tag{7.11}$$

Next we deal with (7.9). Note $\varphi_{\gamma,d}$ is an entire holomorphic function satisfying

$$|\varphi_{\gamma,d}(\zeta)| = |\gamma\varphi(\gamma\zeta)| \le C_M\gamma\exp\left((\gamma-d)\mathrm{Im}\,\zeta\right)(1+|\gamma\zeta|)^{-M}$$

for any N, $\mathrm{Im}\,\zeta \ge 0$ by the Paley-Wiener theorem [9].

Since w is entire of order 3, the Jensen formula gives $N_1(r) = \#\{\lambda_j : |\lambda_j| \le r\} \le Cr^3$. Then for $\kappa > 0$,

$$\begin{aligned} &\left|\sum_{\{\mathrm{Im}\,\lambda_j \ge \kappa\}} \varphi_{\gamma,d}(\lambda_j)\right| \\ &\le C\gamma e^{(\gamma-d)\kappa}\int_0^\infty (1+\gamma r)^{-M}r^2dr && (7.12) \\ &\le C\gamma^{-2}e^{(1-d)\kappa}. && (7.13) \end{aligned}$$

Now assume

$$N(\kappa, r) = \#\left\{\lambda_j : |\mathrm{Re}\,\lambda_j| \le r,\ \mathrm{Im}\,\lambda_j < \kappa\right\} \le C_\epsilon(\kappa)r^{1-\epsilon}$$

for some constant $C_\epsilon(\kappa)$. Then

$$\left| \sum_{\{\operatorname{Im}\lambda_j < \kappa\}} \varphi_{\gamma,d}(\lambda_j) \right| \leq C\gamma \int_0^\infty (1+\gamma r)^{-M} N(\kappa, dr) \leq C\gamma^\epsilon. \tag{7.14}$$

Combining (7.12) and (7.14) we get that (7.10) is bounded from above by

$$C\gamma^{-2} e^{(1-d)\kappa} + C\gamma^\epsilon.$$

Hence we have the inequality

$$C^{-1} d e^{-\delta d} \leq C\gamma^{-2} e^{(1-d)\kappa} + C\gamma^\epsilon$$

which yields a contradiction once we set $\gamma = e^{-\beta d}$ with $\beta > \delta/\epsilon$, and $C\epsilon^{-1} > \kappa > \delta + 2(\delta/\epsilon)$. □

7.2. **Lower Bounds in Logarithmic Neighbourhoods.** In this section, we use a theorem from [15] to get improved lower bounds in logarithmic neighbourhoods of the real axis. To this end, let $\Lambda = \{\lambda_j\}$ be the set of zeros for $w(s) = Z(is + \delta)$, and let $\Lambda_\rho = \{\lambda_j : \operatorname{Im}\lambda_j < \rho \log |\lambda_j|\}$. Let

$$\begin{aligned} N_1(r) &= \#\{\lambda_j : |\lambda_j| \leq r\}, \\ N_\rho(r) &= \#\{\lambda_j \in \Lambda_\rho : |\lambda_j| \leq r\}. \end{aligned}$$

We know $N_1(r) = \mathcal{O}(r^3)$ from Proposition 3.1. We use a slightly different test function for this development. Let $\varphi \in C_0^\infty(\mathbb{R})$ satisfy $\operatorname{supp}(\varphi) \subset [-1,1]$, $\varphi(0) = 1$, $\widehat{\varphi}(\zeta) \geq 0$ for all $\zeta \in \mathbb{R}$ and for $d > 1$, $\gamma < 1$, set $\varphi_{\gamma,d} = \varphi(\gamma^{-1}(t-d))$. We will need the following lemma, taken directly from [15]:

Lemma 7.2. *Suppose $\{\lambda_j\} \subset \mathbb{C}$ is a sequence of points such that $u(t) := \sum_j e^{it\lambda_j}$ belongs to $\mathcal{D}'(\mathbb{R}_+)$. Suppose for some $k \in \mathbb{R}$ and fixed $d > 0$ there is a constant $0 < C < \infty$ such that*

$$\left| \widehat{(\varphi u)}(\lambda) \right| > \left(C^{-1} - o(1)\right) |\lambda|^k, \qquad \lambda \to \infty \tag{7.15}$$

for every $\varphi \in C_0^\infty(\mathbb{R}_+)$ with sufficiently small support such that $\varphi(d) = 1$. Then for every sufficiently small $\epsilon > 0$ and $\rho > (n-k)/(d-\epsilon^2)$ we have:
a) If $k \geq 0$, then

$$N_\rho(r) > \left(\frac{1}{C\pi(k+1)} - o(1) \right) r^{k+1} \tag{7.16}$$

and moreover

$$\sum_{\lambda_j \in \Lambda_\rho,\ |\operatorname{Re}\lambda_j| \leq r} e^{-(d-\epsilon)\operatorname{Im}\lambda_j} > \left(\frac{1}{C\pi(k+1)} - o_\epsilon(1) \right) r^{k+1}.$$

b) If $k < 0$, then for every $\eta > 0$, there is $r(\eta) > 0$ such that

$$N_\rho(r) > r^{1-\eta} \quad \text{if} \quad r > r(\eta).$$

For a proof see [15]. We will use the first part of the lemma to deduce the following "honest" linear lower bound:

Corollary 6. *For $\varphi_{\gamma,d}$ and ρ as above,*

$$N_\rho(r) \geq (C^{-1} - o_\gamma(1)) r. \tag{7.17}$$

Proof. Using the distribution identity (7.1), we see u_1 is of the correct form to apply Lemma 7.2. It remains then only to verify that (7.15) holds with $k = 0$. If $f(z)$ is hyperbolic expanding rational, we can replace f with an appropriate iterate so that $f'(z) > 1$ on $\mathcal{J}$. Then $n \log(A) \leq L_n(z) \leq n \log(B)$ for all n, $f^n(z) = z$, where $A = \min_{\mathcal{J}} f'(z)$ and $B = \max_{\mathcal{J}} f'(z)$. Since there are precisely m^n discrete orbits for each n and the $L_n(z) \to \infty$, if we fix n we can find γ_n small enough and ℓ close to $n \log(AB)^{1/2}$ so that $\ell = L_n(z)$ for at least one orbit and

$$\varphi_{\gamma,\ell}(L_n(z)) = \begin{cases} 1 & \text{if } L_n(z) = \ell \\ 0 & \text{otherwise.} \end{cases}$$

Then for this $\varphi_{\gamma,d}$, we calculate for u_1, u_2 defined above:

$$\begin{aligned} |\widehat{(u_1\varphi_{\gamma,d})}(\lambda)| &= |\widehat{(u_2\varphi_{\gamma,d})}(\lambda)| \\ &= \left| \sum_n \frac{1}{n} \sum_{f^n(z)=z} \left(\frac{L_n(z) e^{-\delta L_n(z)} \varphi_{\gamma,d}(L_n(z))}{|\det(I - d\,(f^n)^{-1}(z))|} e^{-iL_n(z)\lambda} \right) \right| \\ &= \sum_n \frac{1}{n} \sum_{\substack{f^n(z)=z \\ L_n(z)=\ell}} \frac{\ell e^{-\delta \ell}}{|\det(I - d\,(f^n)^{-1}(z))|} \qquad (7.18) \\ &> C^{-1} \qquad (7.19) \end{aligned}$$

with (7.18) and (7.19) holding because the sums are finite and all terms are positive. Thus u_1 satisfies the hypotheses of Lemma 7.2 with $k = 0$ and (7.16) gives (7.17). □

8. Final Comments

Experimental evidence in [16] suggests Conjecture 5 is true. However, as is common with this type of estimates, sharp lower bounds have remained elusive. In order to illustrate the subtlety of this question, we will look at the following example.

Assume for simplicity that $f(z) = z^2 + c$ for c real, $c < -2$, and that A/B is irrational, with $1 < A = \min_{\mathcal{J}} |f'(z)| < B = \max_{\mathcal{J}} |f'(z)|$ as before. Then $\mathcal{J}$ is a Cantor-like set in the real line and all the proofs above go through by complexifying to $\mathbb{C}$ instead of $\mathbb{C}^2$. In the proof of Corollary 6, we stated that the distribution of the $L_n(z)$s is Gaussian with concentration at $\log(AB)^{1/2}$. This suggests a simple model for the zeta function. With A and B as above, we model the distribution of the $L_n(z)$s in the following fashion. We write $L_n(z) = kl_1 + (n-k)l_2$ with multiplicity $\binom{n}{k}$, where we have set $l_1 = \log A$ and $l_2 = \log B$. Using (5.2) as a basis, we calculate

$$
\begin{aligned}
-\sum \frac{1}{n} \sum_{f^n(z)=z} \frac{\exp(-sL_n(z))}{(1-\exp(-L_n(z)))} &= \\
&= -\sum_n \frac{1}{n} \sum_{f^n(z)=z} \sum_k \exp(-(s+k)L_n(z)) \\
&= -\sum_n \frac{1}{n} \sum_k \sum_m \binom{n}{m} (\exp(-(s+k)l_1))^m (\exp(-(s+k)l_2))^{n-m} \\
&= -\sum_n \frac{1}{n} \sum_k (\exp(-(s+k)l_1) + \exp(-(s+k)l_2))^n \\
&= -\sum_k \log\left(1 - e^{-(s+k)l_1} - e^{-(s+k)l_2}\right) \\
&= \log \prod_k \left(1 - e^{-(s+k)l_1} - e^{-(s+k)l_2}\right)
\end{aligned}
$$

so we set

$$
\widetilde{Z}(s) = \prod_k \left(1 - A^{-(s+k)} - B^{-(s+k)}\right). \tag{8.1}
$$

This model shares some important features with $Z(s)$. First, it has one zero at $s = \delta$, where δ, solving $A^\delta + B^\delta = 1$, is the "dimension". Second, it is easy to see that, since A/B is irrational, there are no other zeros on $\operatorname{Re} s = \delta$. However, if $\operatorname{Re} s > -C$, we can take

$$
\widetilde{Z}(s) \sim C \prod_{k=0}^{K} \left(1 - A^{-(s+k)} - B^{-(s+k)}\right)
$$

for some K. Then as $|s| \to \infty$, $\widetilde{Z}(s) \sim Ce^{C|s|}$, whence the number of zeros in $\{\operatorname{Re} s > -C, |\operatorname{Im} s| \le r\}$ grows linearly.

Appendix A

The following lemma is widely known in the literature, but we include a proof of the general result here for completeness.

Lemma A.1. *Suppose $\Omega \subset \mathbb{R}^2$ is an open, bounded, domain, and $f : \Omega \to \Omega$ is an analytic contraction obtained from a holomorphic function on $\mathbb{C}$ identified with $\mathbb{R}^2$. Let $\tilde{f} : \widetilde{\Omega} \to \widetilde{\Omega}$ denote the extension of f to a holomorphic contraction on a bounded, domain $\widetilde{\Omega} \subset \mathbb{C}^2$. Suppose z_1 is the unique fixed point of $\tilde{f}$. Then the pullback operator $\tilde{f}^* : H^2(\widetilde{\Omega}) \to H^2(\widetilde{\Omega})$ has trace*

$$
\operatorname{tr} \tilde{f}^* = \frac{1}{\left|\det\left(I - d\tilde{f}(z_1)\right)\right|}. \tag{A.1}
$$

We first prove this result in the case $\widetilde{\Omega}$ is a ball.

Lemma A.2. *Suppose $f : B_{\mathbb{R}^2}(z_0, r) \to B_{\mathbb{R}^2}(z_0, r')$ is an analytic contraction obtained from a holomorphic function on $\mathbb{C}$ identified with $\mathbb{R}^2$, and let $\tilde{f} : B_{\mathbb{C}^2}(z_0, r) \to B_{\mathbb{C}^2}(z_0, r')$ be the holomorphic extension of f to a neighbourhood of $B_{\mathbb{R}^2}(z_0, r)$ in $\mathbb{C}^2$.*

If z_1 is the unique fixed point of $\tilde{f}$, then the pullback by $\tilde{f}$, $\tilde{f}^* : H^2(B_{\mathbb{C}^2}(z_0, r)) \to H^2(B_{\mathbb{C}^2}(z_0, r))$ has trace

$$\operatorname{tr} \tilde{f}^* = \frac{1}{\left|\det\left(I - d\,\tilde{f}(z_1)\right)\right|}.$$

Proof. Without loss of generality, $z_0 = 0$, and $\tilde{f} : B_{\mathbb{C}^2}(0,1) \to B_{\mathbb{C}^2}(0,\rho)$ for some $\rho \in (0,1)$. Since the group $SU_{\mathbb{C}}(2,1)$ acts transitively on the unit ball in $\mathbb{C}^2$, by composing with appropriate Möbius transformations we may also assume $z_1 = 0$ (see [1]). We first consider $f : B_{\mathbb{R}^2}(0,1) \to B_{\mathbb{R}^2}(0,\rho)$. The assumption that f is obtained from a holomorphic function on $\mathbb{C}$ means for $z \in \mathbb{C}$,

$$d\,f(0)z = (a+ib)(x+iy) = (ax - by) + i(bx + ay)$$

for some $a, b \in \mathbb{R}$. But this implies $d\,f_{\mathbb{R}^2}(0)$ and hence $d\,\tilde{f}(0)$ has the very special form

$$d\,\tilde{f}(0) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a, b \in \mathbb{R}.$$

Thus $d\,\tilde{f}(0)$ is always diagonalizable. Note then that if

$$d\,\tilde{f}(0) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 - bz_2 \\ bz_1 + az_2 \end{pmatrix},$$

then the change of variables

$$\begin{aligned} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} &= A \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \\ &= \frac{1}{a^2+b^2} \begin{pmatrix} a(a-ib) & -b(a-ib) \\ b(a+ib) & a(a+ib) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned} \tag{A.2}$$

makes $d\,\tilde{f}(0)$ diagonal, and further, $\det A = 1$.

We have an orthonormal basis for $H^2(B_{\mathbb{C}^2}(0,1))$ in the form $\{c_\alpha z^\alpha\}_{\alpha \in \mathbb{N}^2}$ for constants c_α. We can use the Bergman kernel to write the kernel for the pullback operator on $H^2(B_{\mathbb{C}^2}(0,1))$,

$$K_{\tilde{f}^*}(z,s) = \sum_\alpha |c_\alpha|^2 (\tilde{f}(z))^\alpha \overline{s}^\alpha,$$

so that for each $u \in H^2(B_{\mathbb{C}^2}(0,1))$,

$$\tilde{f}^* u(z) = \int_{B_{\mathbb{C}^2}(0,1)} K_{\tilde{f}^*}(z,s) u(s) dm(s).$$

Here $dm(s)$ denotes the usual Lebesgue measure on $\mathbb{C}^2$. We will use the change of variables (A.2) and the fact that $\tilde{f}^*$ is trace class to exchange the integral and sum in the following to get:

$$\begin{aligned}
\operatorname{tr} \tilde{f}^* &= \int_{B_{\mathbb{C}^2}(0,1)} K_{\tilde{f}^*}(z,z) dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 (\tilde{f}(z))^\alpha \overline{z}^\alpha dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 (d\,\tilde{f}(0) z + \mathcal{O}(|z|^2))^\alpha \overline{z}^\alpha dm(z) \\
&= \int_{B_{\mathbb{C}^2}(0,1)} \sum_\alpha |c_\alpha|^2 \left(\left(\begin{array}{cc} a+ib & 0 \\ 0 & a-ib \end{array} \right) w + \mathcal{O}(|w|^2) \right)^\alpha \overline{w}^\alpha dm(w) \\
&= \sum_\alpha (a+ib)^{\alpha_1} (a-ib)^{\alpha_2} \\
&= \left| \det(I - d\,\tilde{f}(0)) \right|^{-1}.
\end{aligned}$$

□

Proof of Lemma A.1. Let B be the largest open ball with center at z_1 so that $B \subset \widetilde{\Omega}$. Since $\tilde{f}$ is a contraction and we can always replace f with an appropriate iterate if necessary, we may assume without loss of generality that $\tilde{f}(\widetilde{\Omega}) \subset B$. Now suppose u is a generalized eigenfunction of $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$ with nonzero eigenvalue λ. That is,

$$\left(\tilde{f}^* - \lambda\right)^k u = 0, \quad \text{but} \quad \left(\tilde{f}^* - \lambda\right)^{k-1} u \neq 0 \tag{A.3}$$

for some $k \in \mathbb{Z}_+$ and $\lambda \neq 0$. We claim u can be extended to an eigenfunction of $\tilde{f}^*$ acting on $H^2(B)$ with the same eigenvalue. Indeed, if (A.3) holds, we have

$$\left(\tilde{f}^* - \lambda\right)^k u = \left[\sum_{j=0}^k \binom{k}{j} (-1)^j \lambda^j \left(\tilde{f}^*\right)^{k-j} \right] u = 0,$$

which motivates setting

$$\tilde{u} := (-1)^{k+1} \lambda^{-k} \left[\sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \left(\tilde{f}^*\right)^{k-j} \right] u. \tag{A.4}$$

The lowest order pullback on the right hand side of (A.4) is order 1, and since $\tilde{f}(B) \subset \tilde{f}(\widetilde{\Omega})$, $\tilde{u}$ is in $H^2(B)$. As $(\tilde{f}^* - \lambda)$ commutes with $\lambda^j (\tilde{f}^*)^{k-j}$, we have

$$\left(\tilde{f}^* - \lambda\right)^k \tilde{u} = 0, \quad \text{but} \quad \left(\tilde{f}^* - \lambda\right)^{k-1} \tilde{u} \neq 0.$$

Lastly, if u is a generalized eigenfunction of $\tilde{f}^*$ acting on $H^2(B)$, clearly the restriction of u to $\tilde{f}(\widetilde{\Omega})$ is a generalized eigenfunction with the same eigenvalue for $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$. Then the trace of $\tilde{f}^*$ acting on $H^2(\tilde{f}(\widetilde{\Omega}))$ and $H^2(B)$ are the same and we can apply Lemma A.2 to get (A.1). □

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