QUANTUM MONODROMY AND NON-CONCENTRATION NEAR A CLOSED SEMI-HYPERBOLIC ORBIT

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Abstract. For a large class of semiclassical operators $P(h)-z$ which includes Schrödinger operators on manifolds with boundary, we construct the Quantum Monodromy operator $M(z)$ associated to a periodic orbit γ of the classical flow. Using estimates relating $M(z)$ and $P(h)-z$, we prove semiclassical estimates for small complex perturbations of $P(h)-z$ in the case γ is semi-hyperbolic. As our main application, we give logarithmic lower bounds on the mass of eigenfunctions away from semi-hyperbolic orbits of the associated classical flow.

As a second application of the Monodromy Operator construction, we prove if γ is an elliptic orbit, then $P(h)$ admits quasimodes which are well-localized near γ.

1. Introduction

1.1. **Statement of Results.** To motivate our general results, we first present a few applications. Suppose (X,g) is a compact Riemannian manifold with or without boundary. Let $-\Delta_g$ be the Laplace-Beltrami operator on X and assume u solves the eigenvalue problem

$$-\Delta_g u = \lambda^2 u, \ \ \|u\|_{L^2(X)} = 1.$$

Assume γ is a closed semi-hyperbolic geodesic satisfying either $\gamma \cap \partial X = \emptyset$, or the reflection at the boundary is transversal. Then if U is a sufficiently small neighbourhood of γ, we prove

$$\int_{X\setminus U} |u|^2 dx \geq \frac{C}{\log|\lambda|}, \ \ |\lambda| \to \infty. \tag{1.1}$$

From [Chr1, Chr1a], we have an application to sub-exponential decay of L^2 energy for the damped wave equation: suppose $a(x)$ is positive outside of U and u satisfies

$$\begin{cases} \left(\partial_t^2 - \Delta + 2a(x)\partial_t\right) u(x,t) = 0, & (x,t) \in X \times (0,\infty) \\ u(x,0) = 0, \quad \partial_t u(x,0) = f(x). \end{cases}$$

Then for every $\epsilon > 0$ there exists $C > 0$ such that

$$\|\partial_t u\|^2_{L^2(X)} + \|\nabla u\|^2_{L^2(X)} \leq C e^{-t^{1/2}/C} \|f\|^2_{H^\epsilon(X)}.$$

In addition, we have two dispersive type estimates from [Chr2]. The first is a local smoothing estimate for the Schrödinger equation. Suppose X is a non-compact manifold which is asymptotically Euclidean, and there is a hyperbolic closed geodesic $\gamma \subset U \Subset X$. Then for every $\epsilon > 0$ there exists $C > 0$ such that

$$\int_0^T \left\| \rho_s e^{it(\Delta_g - V(x))} u_0 \right\|^2_{H^{1/2-\epsilon}(X)} dt \leq C \|u_0\|^2_{L^2(X)},$$

where $\rho_s \in \mathcal{C}^\infty(M)$ satisfies

$$\rho_s(x) \equiv \langle d_g(x, x_0) \rangle^{-s}$$

for x_0 fixed and x outside a compact set, and $V \in \mathcal{C}^\infty(M)$, $0 \leq V \leq C$ satisfies

$$|\nabla V| \leq C \langle \text{dist}\,(x, x_0) \rangle^{-1-\delta}$$

for some $\delta > 0$.

The second dispersive estimate is a sub-exponential local energy decay rate for solutions to the wave equation in odd dimensions $n \geq 3$. Suppose X is a non-compact Riemannian manifold which is Euclidean outside a compact set, and suppose u solves

$$\begin{cases} (-D_t^2 - \Delta_g + V(x))u(x,t) = 0, & X \times [0, \infty) \\ u(x,0) = u_0 \in H^1(X), \ D_t u(x,0) = u_1 \in L^2(X) \end{cases}$$

for u_0 and u_1 smooth, compactly supported, where $V \in \mathcal{C}^\infty(M)$ satisfies

$$\exp(-\text{dist}\,_g(x, x_0)^2)V = o(1).$$

Let $\psi \in \mathcal{C}^\infty(X)$ satisfy

$$\psi \equiv \exp(-\text{dist}\,_g(x, x_0)^2)$$

for x outside a compact set and x_0 fixed. Then for every $\epsilon > 0$ there exists $C > 0$ such that

$$\begin{aligned} &\|\psi \partial_t u\|^2_{L^2(X)} + \|\psi u\|^2_{H^1(X)} \\ &\quad \leq C e^{-t^{1/2}/C} \left(\|\partial_t u(x,0)\|^2_{H^\epsilon(X)} + \|u(x,0)\|^2_{H^{1+\epsilon}(X)} \right). \end{aligned}$$

For the general statement of results, let X be a smooth, compact manifold. In this introduction, we state the Main Theorem only in the case $\partial X = \emptyset$. The case with boundary will be considered in §2. We take $P(h) \in \Psi_h^{k,0}$ for $k \geq 1$ and assume $P(h)$ is of real principal type. That is, if $p = \mathcal{C}^\infty(T^*X)$ is the principal symbol of $P(h)$, then p is real-valued, independent of h. Assume $p^{-1}(E)$ is a smooth, compact hypersurface and $dp(x, \xi) \neq 0$ for energies E near 0. We assume p is elliptic outside of a compact subset of T^*X: there exists $C > 0$ such that

$$|\xi| > C \implies p(x, \xi) \geq \langle \xi \rangle^k / C.$$

Let $\Phi_t = \exp tH_p$ be the Hamiltonian flow of p, and suppose Φ_t has a closed, semi-hyperbolic orbit $\gamma \subset \{p = 0\}$ of period T. The assumption that γ be semi-hyperbolic means if N is a Poincaré section for γ and $S : N \to S(N)$ is the Poincaré map, then the linearization of S, $dS(0)$, is nondegenerate and has at least one eigenvalue off the unit circle. For the eigenvalues of modulus 1 we also require the following nonresonance assumption for the Poincaré map at the energy 0:

$$\begin{cases} \text{if } e^{\pm i\alpha_1}, e^{\pm i\alpha_2}, \ldots, e^{\pm i\alpha_k} \text{ are eigenvalues of modulus 1, then} \\ \alpha_1, \alpha_2, \ldots, \alpha_k \text{ are independent over } \pi\mathbb{Z}. \end{cases} \tag{1.2}$$

Our Main Theorem is that for a family of eigenfunctions $u(h)$ for $P(h)$,

$$P(h)u(h) = E(h)u(h), \ \ E(h) \to 0 \text{ as } h \to 0,$$

$u(h)$ has its mass concentrated away from γ. This is made precise in the following theorem.[1]

Main Theorem. *Let $A \in \Psi_h^{0,0}(X)$ be a pseudodifferential operator whose principal symbol is* 1 *near γ and* 0 *away from γ. There exist constants $h_0 > 0$ and $C > 0$ such that*

$$\|u\| \leq C \frac{\log(1/h)}{h} \|P(h)u\| + C\sqrt{\log(1/h)} \|(I - A)u\|$$

uniformly in $0 < h < h_0$, where the norms are L^2 norms on X. In particular, if $u(h)$ satisfies

$$\begin{cases} P(h)u(h) = \mathcal{O}(h^\infty); \\ \|u(h)\|_{L^2(X)} = 1, \end{cases}$$

$$\|(I - A)u\|_{L^2(X)} \geq \frac{1}{C} \left(\log(1/h) \right)^{-1/2}, \quad 0 < h < h_0.$$

Remark 1.1. In §2 we assume $P(h) \in \mathrm{Diff}_h^2(X)$ is a differential operator on X and that ∂X is noncharacteristic with respect to the principal symbol of $P(h)$. Then a similar conclusion to the Main Theorem holds (see Main Theorem' in that section).

Remark 1.2. In §9, we give a partial converse to the Main Theorem in Theorem 7. That is, the techniques of the proof of the Main Theorem are used to show if the periodic orbit γ is elliptic, then $P(h)$ admits quasimodes which are well-localized to γ. This result is well-known in the literature (see, for example, [Ral], [CdV], and [CaPo] and the references cited therein), however to our knowledge the use of the Quantum Monodromy operator to construct these quasimodes is new.

Remark 1.3. We interpret the assumption that γ be semi-hyperbolic as follows: if γ were fully hyperbolic, then we know from [Chr1] that the mass of eigenfunctions cannot concentrate very rapidly (as the eigenvalue goes to infinity) near γ. However, we know from Theorem 7 below that if γ were fully elliptic, then there are at least *approximate* eigenfunctions with *all* their mass concentrated near γ. The rough heuristic is that if γ is hyperbolic in at least one direction in phase space and a sequence of eigenfunctions were to concentrate rapidly near γ, then they would have to do so in a lower dimensional set - a clear violation of the uncertainty principle. Of course the proof is quite a bit more technical. The nonresonance assumption on the elliptic eigenvalues is necessary in order to assume there are no other periodic orbits in a neighbourhood of γ in the set $\{p = 0\}$. This in turn implies (see, for instance [AbMa, Theorem 28.5]) that there is a range of energy surfaces $\{p = z\}$ near $z = 0$ in which there is an isolated periodic semi-hyperbolic orbit. The family of these orbits is diffeomorphic to a cylinder.

Remark 1.4. The estimates in this work are all microlocal in nature, hence we lose nothing by assuming X is compact. In order to apply these estimates in the case of non-compact manifolds, we assume P is elliptic outside a compact set (that is, $|p| \geq \langle \xi \rangle^k / C$ for $|(x, \xi)| \geq C$) and the geometry is non-trapping outside of a compact submanifold and then apply our results there. See [Chr2] for more on this.

[1] In the interest of length, for background, definitions, and standard material referenced in this paper, we refer the reader to [Chr1, §2] and the references cited therein, as well as to the excellent online book of Evans-Zworski [EvZw].

The Main Theorem is the similar to [Chr1, Main Theorem] with three generalizations, namely that we no longer assume the linearized Poincaré map has no negative eigenvalues, we allow some eigenvalues of modulus 1, and in §2 we allow γ to reflect transversally off ∂X with some extra assumptions on $P(h)$. This allows study of, for example, billiard problems in any dimension. The problems encountered in [Chr1] with these cases come from attempting to put p into a normal form in a neighbourhood of $\gamma \subset T^*X$.

The motivation for the proof in this paper is to reduce the problem of studying the resolvent $(P-z)^{-1}$ in a microlocal neighbourhood of γ to studying a related operator on the Poincaré section N.

If we identify N with $T_0^*N \simeq \mathbb{R}^{2n-2}$ near 0, we are led to study operators acting on $L^2(V)$, where $V \subset \mathbb{R}^{n-1}$. In the course of this work, we will see the relevent object of study is the Quantum Monodromy operator $M(z) : L^2(V) \to L^2(V)$. By setting up a Grushin problem in a neighbourhood of

$$\gamma \times (0,0) \subset T^*X \times T^*\mathbb{R}^{n-1},$$

and using the microlocal inverse constructed by Sjöstrand-Zworski in [SjZw1], we will see it is sufficient to bound $\|I - M(z)\|_{L^2(V) \to L^2(V)}$ from below. This will result in the following theorem.

Theorem 1. *Suppose $P \in \Psi_h^{k,0}$ is a semiclassical pseudodifferential operator of real principal type satisfying all of the assumptions of the introduction, and assume $\gamma \cap \partial X = \emptyset$. Then there exist positive constants C, c_0, h_0, ϵ_0, and a positive integer N such that for $0 < h < h_0$, $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$, if $u \in L^2(X)$ has h-wavefront set sufficiently close to γ, then*

$$\|(P-z)u\|_{L^2(X)} \geq C^{-1} h^N \|u\|_{L^2(X)}. \tag{1.3}$$

Theorem 1 allows us to add a complex absorption term of order h supported away from γ. Let $a \in \mathcal{C}^\infty(T^*X)$ equal 0 in a neighbourhood of γ and 1 away from γ, and define

$$Q(z)u = P(h) - z - ihCa^w \tag{1.4}$$

for a constant $C > 0$ to be chosen later. Then a semiclassical adaptation of the "three-lines" theorem from complex analysis, will allow us to deduce the following estimate.

Theorem 2. *Suppose $Q(z)$ is given by* (1.4)*, and*

$$z \in [-\epsilon_0/2, \epsilon_0/2] + i(-ch/\log(1/h), ch/\log(1/h))$$

for $\epsilon_0, c > 0$ sufficientlyl small. Then there is $h_0 > 0$ and $0 < C < \infty$ such that for $0 < h < h_0$,

$$\left\|Q(z)^{-1}\right\|_{L^2(X) \to L^2(X)} \leq C \frac{\log(1/h)}{h}. \tag{1.5}$$

As an intermediate step to proving Theorem 2, we first prove a similar statement about an operator with a larger absorbing term. Let $W \in \mathcal{C}^\infty(T^*X)$, $W \equiv 0$ in a small neighbourhood of γ and $W \equiv 1$ away from γ. We define a perturbed family of operators

$$\widetilde{Q}(z) = P(h) - z - iW^w. \tag{1.6}$$

We have the following Theorem.

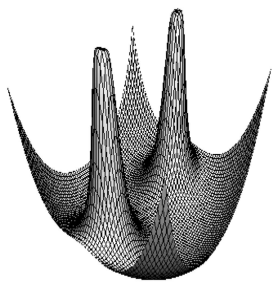

FIGURE 1. A confining potential $V(x)$ with two bumps at the lowest energy level $E < 0$.

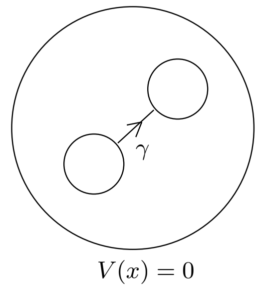

FIGURE 2. The level set $V(x) = 0$ and the closed hyperbolic orbit γ reflecting off the "soft" boundary.

Theorem 3. *Suppose $\widetilde{Q}(z)$ is as above, $z \in [-\epsilon_0/2, \epsilon_0/2]$, and $W \equiv 1$ outside a sufficiently small neighbourhood of γ. Then there is $h_0 > 0$ and $0 < C < \infty$ such that for $0 < h < h_0$,*

$$\left\| \widetilde{Q}(z)^{-1} \right\|_{L^2(X) \to L^2(X)} \leq C \frac{\log(1/h)}{h}. \tag{1.7}$$

If $\varphi \in \mathcal{C}_c^\infty(X)$ is supported away from γ, then

$$\left\| \widetilde{Q}(z)^{-1} \varphi \right\|_{L^2(X) \to L^2(X)} \leq C \frac{\sqrt{\log(1/h)}}{h}. \tag{1.8}$$

Theorem 2 follows from Theorem 3 using the control theory arguments exactly as in [Chr1a].

1.2. **Examples.** There are many examples in which the hypotheses of the theorem are satisfied, the simplest of which is the case in which $p = |\xi|^2 - E(h)$ for $E(h) > 0$. Then the Hamiltonian flow of p is the geodesic flow, so if the geodesic flow has a closed semi-hyperbolic orbit, there is non-concentration of eigenfunctions, $u(h)$, for the equation

$$-h^2 \Delta u(h) = E(h) u(h).$$

Another example of such a p is the case $p = |\xi|^2 + V(x)$, where $V(x)$ is a confining potential with two "bumps" or "obstacles" in the lowest energy level (see Figure 1). In the appendix to [Sjö] it is shown that for an interval of energies $V(x) \sim 0$, there is a closed hyperblic orbit γ of the Hamiltonian flow which "reflects" off the bumps (see Figure 2). Complex hyperbolic orbits may be constructed by considering 3-dimensional hyperbolic billiard problems (see, for example, [AuMa, §2]). In addition, Proposition 4.3 from [Chr1] gives a somewhat artificial means of constructing a manifold diffeomorphic to a neighbourhood in $T^*\mathbb{S}^1_{(t,\tau)} \times T^*\mathbb{R}^{n-1}_{(x,\xi)}$ which contains a hyperbolic orbit γ by starting with the Poincaré map γ is to have.

In the appendix, we examine the Riemannian manifold

$$M = \mathbb{R}_x/\mathbb{Z} \times \mathbb{R}_y \times \mathbb{R}_z$$

equipped with the metric

$$ds^2 = \cosh^2 y(2z^4 - z^2 + 1)^2 dx^2 + dy^2 + dz^2,$$

which has a semi-hyperbolic closed geodesic at $y = z = 0$ and two hyperbolic closed geodesics at $y = 0, z = \pm 1/2$. Restricting y and z to compact intervals yields a compact manifold satisfying the hypotheses of the Main Theorem, while the non-compact manifold provides a model for possibly extending the dispersive-type estimates to the semi-hyperbolic case.

1.3. **Organization.** This work is organized as follows. In §2 we first state the versions of Theorems 1, 2, and the Main Theorem in the case of a compact manifold with boundary, then review the classical picture of a closed orbit reflecting transversally off the boundary, and prove a propagation of singularities result at the boundary. §3 gives the definition and basic facts about the Quantum Monodromy operator $M(z)$, while §4 shows how $M(z)$ arises naturally in the context of a Grushin problem. §5 presents the main ideas of the proof of Theorem 1 by considering a model. In §6-7 the proof of Theorem 1 is presented, while the proof of Theorem 2 and the Main Theorem is reserved for §8. Finally, in §9, we show how the Monodromy operator construction can be used to construct well-localized quasimodes if γ is elliptic. In the appendix, we provide a concrete example of a semi-hyperbolic orbit.

1.4. **Acknowledgements.** The author would like to thank Maciej Zworski for much help and support during the writing of this work, as well as Nicolas Burq for suggesting study of the monodromy operator as a means of tackling the boundary problem. He would also like to thank Herbert Koch for suggesting the generalization to semi-hyperbolic orbits, and Michael Hitrik for much help in working out the model case for Theorem 7. The majority of this work was conducted while the author was a graduate student in the Mathematics Department at UC-Berkeley and he is very grateful for the support received while there.

2. Manifolds with Boundary and Propagation of Singularities

In this section, X is a smooth, compact, n-dimensional manifold with boundary. We assume $P \in \mathrm{Diff}^{2,0}_{h,db}$ is a second order differential operator whose principal symbol p is a quadratic form in ξ and ∂X is noncharacteristic with respect to p. We adopt a microlocal viewpoint in which ∂X is identified locally with a noncharacteristic hypersurface $Y \subset \mathbb{R}^n$. Our local model for X near Y is $X = \mathbb{R}^n$ with $Y = \{x \in \mathbb{R}^n : x_1 = 0\}$. We study the boundary value problem

$$\begin{cases} (P - z)u = f \text{ in } X, \\ u = 0 \text{ on } Y, \end{cases} \tag{2.1}$$

in a neighbourhood of a closed bicharacteristic for the flow of H_p reflecting transversally off Y, and for energies z near 0. Our final goal is to describe propagation of singularities at the boundary. First we will prove factorization lemmas and energy estimates near Y, and then prove the main result of this section, which is that the microlocal propagator of $P - z$ can be extended in a meaningful way through the reflections at the boundary. The Main Theorem has the following analogue in the case γ reflects transversally off ∂X (see §2.1 for definitions).

Main Theorem'. *Suppose $P(h) \in \mathit{Diff}_h^2(X)$ and ∂X is noncharacteristic with respect to the principal symbol of $P(h)$. Assume γ makes only transversal reflections with ∂X. Let $A \in \Psi_{h,db}^{0,0}(X)$ be a pseudodifferential operator whose principal symbol is 1 near γ and 0 away from γ. There exist constants $h_0 > 0$ and $C > 0$ such that*

$$\|u\| \leq C\frac{\log(1/h)}{h}\|P(h)u\| + C\log(1/h)\|(I-A)u\|$$

uniformly in $0 < h < h_0$, where the norms are L^2 norms on X. In particular, if $u(h)$ satisfies

$$\begin{cases} P(h)u(h) = \mathcal{O}(h^\infty); \\ \|u(h)\|_{L^2(X)} = 1, \end{cases}$$

$$\|(I-A)u\|_{L^2(X)} \geq \frac{1}{C}\log\left((1/h)\right)^{-1}, \quad 0 < h < h_0.$$

As in §1, Main Theorem' is a consequence of the following versions of Theorems 1 and 2 in the case of a manifold with boundary.

Theorem 1'. *Suppose $P(h) \in \mathit{Diff}_h^2(X)$ and ∂X is noncharacteristic with respect to the principal symbol of $P(h)$. Assume γ makes only transversal reflections with ∂X. Then there exist positive constants C, c_0, h_0, ϵ_0, and a positive integer N such that for $0 < h < h_0$, $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$, if $u \in L^2(X)$ has h-wavefront set sufficiently close to γ, then*

$$\|(P-z)u\|_{L^2(X)} \geq C^{-1}h^N\|u\|_{L^2(X)}. \tag{2.2}$$

As in the introduction, we add a complex absorbing term: let $a^w \in \Psi_{h,db}$ equal 0 in a neighbourhood of γ and 1 away from γ (according to the equivalence relation defined in §2.1), and define as usual

$$Q(z)u = P(h) - z - ihCa^w$$

for a constant $C > 0$ to be chosen later. We have the following version of Theorem 2.

Theorem 2'. *Suppose $Q(z)$ is given as above, and*

$$z \in [-\epsilon_0/2, \epsilon_0/2] + i(-ch/\log(1/h), ch/\log(1/h))$$

for $\epsilon_0, c > 0$ sufficientlyl small. Then there is $h_0 > 0$ and $0 < C < \infty$ such that for $0 < h < h_0$,

$$\left\|Q(z)^{-1}\right\|_{L^2(X)\to L^2(X)} \leq C\frac{\log(1/h)}{h}. \tag{2.3}$$

Just as in the introduction, before proving Theorem 2, we first need an auxiliary theorem. Let $W \in \mathcal{C}^\infty(T^*X)$, $W \equiv 0$ in a small neighbourhood of γ and $W \equiv 1$ away from γ (again using the equivalence relations in §2.1). We define a perturbed family of operators

$$\widetilde{Q}(z) = P(h) - z - iW^w.$$

We have the following Theorem.

Theorem 3'. *Suppose $\widetilde{Q}(z)$ is as above, $z \in [-\epsilon_0/2, \epsilon_0/2]$, and $W \equiv 1$ outside a sufficiently small neighbourhood of γ. Then there is $h_0 > 0$ and $0 < C < \infty$ such that for $0 < h < h_0$,*

$$\left\| \widetilde{Q}(z)^{-1} \right\|_{L^2(X) \to L^2(X)} \leq C \frac{\log(1/h)}{h}. \tag{2.4}$$

If $\varphi \in \mathcal{C}_c^\infty(X)$ is supported away from γ, then

$$\left\| \widetilde{Q}(z)^{-1} \varphi \right\|_{L^2(X) \to L^2(X)} \leq C \frac{\sqrt{\log(1/h)}}{h}. \tag{2.5}$$

Theorem 2 follows from Theorem 3 using the control theory arguments exactly as in [Chr1a].

2.1. **Normally Differential Operators.** In the case X is a smooth manifold with boundary, we define pseudodifferential operators which are differential in the normal direction at the boundary microlocally. For a microlocal definition, it suffices to assume $X = \{x_1 \geq 0\}$ and $\partial X = \{x_1 = 0\}$. Then the algebra of pseudodifferential operators which are normally differential at the boundary is defined by the following:

$$\Psi_{h,db}^{k,m}(X, \Omega_X^{\frac{1}{2}}) = \Big\{ A(x, hD_x) \in \Psi_h^{k,m} : A(x, hD_x) = \sum_{j=0}^{k} A_j(x, hD_{x'})(hD_{x_1})^j \Big\}.$$

Suppose $\varphi \in \mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}})$, and $x_0 \in \partial X$. Using local coordinates at the boundary, we write $x_0 = (0, x_0') \in \{x_1 \geq 0\}$. Then $\varphi \in \mathcal{C}^\infty(X, \Omega_X^{\frac{1}{2}})$ means there is a smooth extension $\tilde{\varphi}$ to an open neighbourhood of $x_0 \in \mathbb{R}^n$. For a distribution $u \in \mathcal{D}'(X, \Omega_X^{\frac{1}{2}})$, we extend the notion of $\mathrm{WF}_h(u)$ to a neighbourhood of the boundary. We say $(x_0, \xi_0) = (0, x_0', \xi_0)$ is not in $\mathrm{WF}_h(u)$ if there is a product neighbourhood $(x_0, \xi_0) \in U \times V \subset \mathbb{R}^{2n}$ and a normally differential operator $A \in \Psi_{h,db}^{0,0}(U, \Omega_U^{\frac{1}{2}})$ such that $\sigma_h(A)(x_0, \xi_0) \neq 0$ and

$$Au \in h^\infty \mathcal{C}^\infty((0,1]_h; \mathcal{C}^\infty(U, \Omega_U^{\frac{1}{2}})).$$

Observe if u is smooth,

$$(\mathrm{WF}_h u)|_{\partial X} \subset \mathrm{WF}_h(u|_{\partial X}) \sqcup (\mathrm{supp}\,(u|_{\partial X}) \times N^*(\partial X)).$$

Similarly, using our identification of the h-wavefront set of a pseudodifferential operator as the essential support of its symbol, $A \in \Psi_{h,db}^{0,0}$ with $\sigma_h(A) \neq 0$ at $(0, x_0', \xi_0)$ implies the ξ_1 direction is always contained in the h-wavefront set of A.

We are going to be interested in symbols which are compactly supported in T^*X, so we will need a notion of microlocal equivalence near the boundary which allows us to consider operators which are both normally differential *and* compactly supported in phase space. For this we return to our local coordinates at the boundary. Let $x_0 \in \partial X$, $x_0 = (0, x_0')$, and let $U \times V$ be a product neighbourhood of $(x_0, 0)$ in $\mathbb{R}^{2n}$ such that V is of the form $V = [-\epsilon_0, \epsilon_0]_{\xi_1} \times V_{\xi'}$. By using the rescaling

$$(x_1, \xi_1) \mapsto (x_1/\lambda, \lambda \xi_1),$$

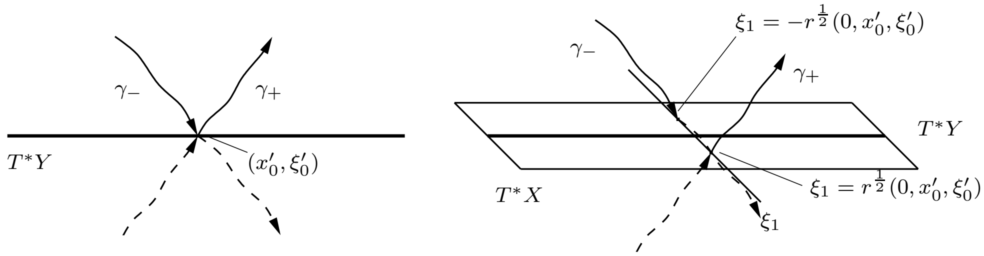

FIGURE 3. The incoming and outgoing bicharacteristics. Observe the *reflection* of γ is continuous in a neighbourhood of H in T^*Y but not in T^*X, while the *transmission* of γ is continuous in both.

the ellipticity of P outside a compact set implies $p \geq C^{-1}$ in

$$(\complement[-\epsilon_0, \epsilon_0]) \times V_{\xi'}.$$

Choose $\psi \in \mathcal{C}^\infty(\mathbb{R})$ satisfying

$$\psi(t) \equiv 1 \text{ for } |t| \leq \epsilon_0, \tag{2.6}$$

$$\psi(t) \equiv 0 \text{ for } |t| \geq 2\epsilon_0. \tag{2.7}$$

We say two semiclassically tempered operators T and T' are microlocally equivalent near $(U \times V)^2$ if for all $A, A' \in \Psi^{0,0}_{h,db}$ satisfying

$$\text{proj}_{(x,\xi')}(\text{WF}_h A) \text{ is sufficiently close to } U \times V_{\xi'},$$

and similarly for A',

$$\psi(P(h))A(T - T')\psi(P(h))A' = \mathcal{O}(h^\infty) : \mathcal{D}'(X) \to \mathcal{C}^\infty(X).$$

In particular, if $A \in \Psi^{0,0}_{h.db}$, we say A is microlocally equivalent to

$$\psi(P(h))A$$

and we will use this identification freely throughout.

2.2. **Propagation of Singularities.** This section is basically a semiclassical adaptation of some of the propagation of singularities results at the boundary presented in [Hor, Chap. 23]. According to [Hor, App. C.5], under the noncharacteristic assumption we can find local symplectic coordinates near Y so that $Y = \{x_1 = 0\}$ and (possibly after a sign change)

$$p(x, \xi) = \xi_1^2 - r(x, \xi'), \;\; \xi' = (\xi_2, \ldots, \xi_n). \tag{2.8}$$

We define the hyperbolic set $H \subset T^*Y$:

$$H := \{(x', \xi') : r(0, x', \xi') > 0\},$$

on the lift of which the characteristic equation has the two roots $\{x_1 = 0, \xi_1 = \pm r(x, \xi')^{\frac{1}{2}}\}$. Thus the Hamiltonian vector field of p,

$$H_p = 2\xi_1 \partial_{x_1} - \partial_{\xi'} r \partial_{x'} + \partial_x r \partial_\xi$$

points from Y into $\{x_1 > 0\}$ or $\{x_1 < 0\}$, respectively, depending on which root of r we choose. We call the corresponding bicharacteristic rays *outgoing* and *incoming* and write γ_+ and γ_- respectively (see Figure 3). We have the following factorization of the operator $P - z$ in our microlocal coordinates.

Lemma 2.1. *There is a factorization of $P - z$ near H:*

$$P - z = (hD_1 - A_-(x, hD'))(hD_1 - A_+(x, hD'))$$

with $A_\pm \in \Psi^{1,0}_{h,db}$ having principal symbol $\pm r^{\frac{1}{2}}$.

Remark 2.2. We remark $r(x, \xi')$ and $A_\pm(x, hD')$ implicitly depend on the energy z, although we don't explicitly note this dependence where no ambiguity can arise.

Proof. We follow the proof for the h independent version of the lemma found in [Hor, Lemma 23.2.8]. Using the coordinates above, the principal symbol of $P - z$ is (2.8). Set $A^1_\pm = \mathrm{Op}\,(\pm r^{\frac{1}{2}})$ so that

$$P - z - (hD_1 - A^1_-)(hD_1 - A^1_+) = R_1(x, hD) \text{ microlocally,}$$

where $\sigma_h R_1 = \mathcal{O}(h)$ is independent of ξ_1.

Suppose now we have $A^j_\pm$ with principal symbols $\pm r^{\frac{1}{2}}$ such that

$$P - z - (hD_1 - A^j_-)(hD_1 - A^j_+) = R_j(x, hD) \text{ microlocally,}$$

where $\sigma_h R_j = \mathcal{O}(h^j)$ is independent of ξ_1. Choose $a^j_-(x, \xi') = \mathcal{O}(h^j)$ satisfying

$$\sigma_h R_j(x, \xi') + 2a^j_-(x, \xi') r^{\frac{1}{2}}(x, \xi') = 0,$$

which we can do since $r^{\frac{1}{2}} > 0$ near H. We will similarly add $a^j_+(x, hD')$ to A^j_+, where $a^j_+ = \mathcal{O}(h^j)$ is determined by the following calculation:

$$\begin{aligned} P - z - (hD_1 - A^j_- - a^j_-(x, hD'))(hD_1 - A^j_+ - a^j_+(x, hD')) = \\ = \quad R_j(x, hD) - a^j_-(x, hD')(hD_1 - A^j_+) - (hD_1 - A^j_-)(a^j_+(x, hD')) \\ + a^j_-(x, hD') a^j_+(x, hD') \text{ microlocally.} \end{aligned}$$

On the level of principal symbol, this yields the requirement that

$$\begin{aligned} \sigma_h R_j(x, \xi') - a^j_-(x, \xi')(\xi_1 - r^{\frac{1}{2}}(x, \xi')) - (\xi_1 + r^{\frac{1}{2}}(x, \xi')) a^j_+(x, \xi') &= \\ = \quad -a^j_-(x, \xi')(\xi_1 + r^{\frac{1}{2}}(x, \xi')) - (\xi_1 + r^{\frac{1}{2}}(x, \xi')) a^j_+(x, \xi') \\ = \quad 0, \end{aligned}$$

which gives $a^j_+(x, \xi') = -a^j_-(x, \xi')$. By induction and Borel's Lemma the argument is complete. ☐

We have also a microlocal factorization $P - z = (hD_1 - \tilde{A}_+)(hD_1 - \tilde{A}_-)$, where the principal symbols of $\tilde{A}_\pm$ are $\pm r^{\frac{1}{2}}$ as in the lemma. Suppose the $\gamma_\pm$ intersect T^*Y at (x'_0, ξ'_0). On γ_- we have $\xi_1 = -r^{\frac{1}{2}}$, so $(hD_1 - \tilde{A}_+)$ is elliptic near γ_-. Then to solve (2.1), we need only solve $(hD_1 - \tilde{A}_-)u = (hD_1 - \tilde{A}_+)^{-1} f = \tilde{f}$.

Lemma 2.3. *Suppose u solves the following Cauchy problem in $\mathbb{R}^n_+$:*

$$\left\{ \begin{array}{l} (hD_1 - \tilde{A}_-)u = \tilde{f}, \;\; x_1 > 0 \\ u|_{x_1 = 0} = \varphi(x'). \end{array} \right. \tag{2.9}$$

Then

$$\begin{aligned} \sup_{0 \le y \le T_0} \|u(y, \cdot)\|_{L^2_{x'}(\mathbb{R}^{n-1} \times \{x_1 = y\})} \le \\ \le \quad C\|\varphi\|_{L^2_{x'}(\mathbb{R}^{n-1} \times \{x_1 = 0\})} + \frac{C_{T_0}}{h} \|\tilde{f}\|_{L^1([0, T_0], L^2(\mathbb{R}^{n-1}))}. \end{aligned} \tag{2.10}$$

Proof. Consider

$$\begin{aligned}\frac{1}{2}\partial_y\|u(y,\cdot)\|^2_{L^2(\mathbb{R}^{n-1}\times\{x_1=y\})} &= \langle\partial_y u,u\rangle_{x'}\\ &= -\frac{\mathrm{Im}}{h}\langle hD_1u,u\rangle_{x'}\\ &\leq \frac{1}{h}\|u(y,\cdot)\|_{L^2_{x'}}\|\tilde f(y,\cdot)\|_{L^2_{x'}}\\ &\leq \frac{1}{4h^2}\|\tilde f(y,\cdot)\|^2_{L^2_{x'}}+\|u(y,\cdot)\|^2_{L^2_{x'}}\end{aligned}$$

which by Gronwall's inequality gives the lemma. □

Recall the semiclassical Sobolev norms $\|\cdot\|_{H^k_h}$ are given by

$$\|u\|_{H^k_h(V)}=\left(\sum_{|\alpha|\leq k}\int_V|(hD)^\alpha u|^2dx\right)^{\frac{1}{2}}.$$

We observe that since P is elliptic outside a compact set and $(hD_1-\tilde A_+)$ is elliptic, replacing $\tilde f$ with f and conjugating $\tilde A_-$ above with the invertible operators $(C+P)^{s/2}$ for sufficiently large $C>0$, we can estimate the L^2 norm of $v=(C+P)^{s/2}u$, and we get the Sobolev estimate

$$\begin{aligned}&\sup_{0\leq y\leq T_0}\|u(y,\cdot)\|_{(H^s_h)_{x'}(\mathbb{R}^{n-1}\times\{x_1=y\})}\leq\\ &\leq\quad C\|\varphi\|_{(H^s_h)_{x'}(\mathbb{R}^{n-1}\times\{x_1=0\})}+\frac{C_{T_0}}{h}\|f\|_{L^1([0,T_0],H^s_h(\mathbb{R}^{n-1}))}.\end{aligned}$$

We are interested in proving the existence of a microlocal solution propagator, hence we assume the wavefront set of f is contained in a compact set K in a neighbourhood of γ_- near Y. We assume as well that K is contained in a single coordinate chart U on which the assumptions of [EvZw, Theorem 10.18] hold. Suppose $K\subset\{T_1<x_1<T_2\}$ and $U\subset\{T_1'<x_1<T_2'\}$.

Proposition 2.4. *There are exactly two microlocal solutions u_i, $i=1,2$ to $(hD_1-\tilde A_-)u=\tilde f$ microlocally near γ_- satisfying*

$$u_1 = 0 \text{ microlocally for } x_1\leq T_1, \tag{2.11}$$

$$u_2 = 0 \text{ microlocally for } x_1\geq T_2. \tag{2.12}$$

Proof. First we prove the proposition in a neighbourhood of $\mathrm{WF}_h\tilde f$. Let $\tilde K$ be the coordinate representation of K. Apply [EvZw, Theorem 10.18] to write $(hD_1-\tilde A_-)$ as hD_1 in these coordinates. We write in the x-projection of this coordinate patch,

$$\begin{aligned}u_1(x) &= \frac{i}{h}\int_{-\infty}^{x_1}\tilde f(y,x')dy,\\ u_2(x) &= -\frac{i}{h}\int_{x_1}^{+\infty}\tilde f(y,x')dy,\end{aligned}$$

which satisfies $(hD_1-\tilde A_-)u=\tilde f$ with (2.11-2.12). Back in the original coordinates on our manifold, set $u_1=0$ for $x_1\leq T_1'$ and $u_2=0$ for $x_1\geq T_2'$. To continue, we will employ the energy estimates in Lemma 2.3. Suppose u is a solution to $(hD_1 - \tilde{A}_-)u = \tilde{f}$. Then for $v \in \mathcal{C}_c^\infty([0,T_0) \times \mathbb{R}^{n-1})$,

$$\int_0^{T_0} \left\langle u, (hD_1 - \tilde{A}_-)v \right\rangle_{x'} dy = \int_0^{T_0} \langle \tilde{f}, v\rangle_{x'} dy + \frac{h}{i}\langle u(0,\cdot), v(0,\cdot)\rangle_{x'},$$

since in the Weyl calculus real symbols are self adjoint. But from the proof of Lemma 2.3, replacing y with $T_0 - y$, we have since $\lim_{y\to T_0} v(y,\cdot) = 0$,

$$\sup_{0\le y\le T_0} \|v(y,\cdot)\|_{L^2_{x'}} \le \frac{C}{h}\int_0^{T_0} \|g\|_{L^2_{x'}} dy,$$

with $g = (hD_1 - \tilde{A}_-)v$. We then have

$$\left| \int_0^{T_0} \langle \tilde{f}, v\rangle_{x'} dy + \frac{h}{i}\langle u(0,\cdot), v(0,\cdot)\rangle_{x'} \right| \le \frac{C_{\tilde{f},u_{0,\cdot}}}{h}\int_0^{T_0} \|g\|_{L^2_{x'}} dy.$$

For $h > 0$ we can extend to $g \in L^2$ the complex-conjugate linear form

$$g \mapsto \int_0^{T_0} \langle \tilde{f}, v\rangle_{x'} dy + \frac{h}{i}\langle u(0,\cdot), v(0,\cdot)\rangle_{x'}$$

by the Hahn-Banach Theorem. Thus by the Riesz Representation Theorem, for $\tilde{f} \in \mathcal{C}^\infty$ with sufficiently small wavefront set, we can find $u \in \mathcal{C}^\infty([0,T_0), L^2_{x'})$ satisfying $(hD_1 - \tilde{A}_-)u = \tilde{f}$.

For the uniqueness given by the conditions (2.11-2.12), note that if $f = 0$ and $u(0,\cdot) = 0$ in (2.10), u is zero. Replacing x_1 by $T_0 - x_1$ we get the backwards uniqueness result. □

Since u_1 is supported in the forward direction along the bicharacteristic γ_-, we refer to u_1 and u_2 as the *forward* and *backward* solutions respectively. Let $u_- = u_1$ be the forward solution along the incoming bicharacteristic γ_-. So far we have proved the solution u_- satisfies $(P - z)u_- = f$ near γ_-, $u_- = 0$ microlocally for x_1 larger than the support of f, and u_- restricted to the boundary is controlled by h^{-1} in L^2 if the wavefront set of f is sufficiently small.

The same energy method techniques can be used to solve the problem

$$\begin{cases} (P-z)u_+ = 0, & \text{in } X, \\ u_+|_Y = u_-|_Y \end{cases} \tag{2.13}$$

near γ_+ so that $u = u_- - u_+$ solves (2.1).

Corollary 2.5. *If $f \in H_h^\infty$ has sufficiently small wavefront set and u solves* (2.1)*, then*

$$u \in C^1([0,T_0], H_h^s(\mathbb{R}^{n-1}))$$

for every s. In particular, $u(y,\cdot) \in \mathcal{C}^\infty(\mathbb{R}^{n-1} \times \{x_1 = y\})$ for each fixed $y \in [0,T_0]$.

In order to describe propagation of singularities near the boundary, we first need the following lemma.

Lemma 2.6. *Let γ_+ be an interval on the outgoing bicharacteristic with one endpoint at $(0, x_0', r(0,x_0',\xi_0')^{\frac{1}{2}}, \xi_0')$. Then there is a pseudodifferential operator $Q(x,hD') \in \Psi^{0,0}_{h,db}$ which satisfies*

(i) $\sigma_h(Q) = 0$ microlocally outside a neighbourhood of

$$\{(x,\xi') : (x, r(x,\xi')^{\frac{1}{2}}, \xi') \in \gamma_+\},$$

(ii) Q is noncharacteristic at (x_0', ξ_0'), and
(iii) $[Q(x, hD'), hD_1 - A_+(x, hD')] = 0$ microlocally near γ_+.

Proof. The principal symbol of the commutator $[Q(x, hD'), hD_1 - A_+(x, hD')]$ is

$$-ih\{\sigma_h Q(x, \xi'), \xi_1 - r^{\frac{1}{2}}(x, \xi')\} = ih(\partial_{x_1} - H_{r^{\frac{1}{2}}})\sigma_h Q.$$

First we solve the Cauchy problem

$$\left\{ \begin{array}{l} \left(\partial_{x_1} - H_{r^{\frac{1}{2}}}\right) Q_0 = 0, \ \ (x, \xi) \in T^*X \\ Q_0 = q_0, \ \ x_1 = 0 \end{array} \right. \tag{2.14}$$

so that Q_0 is constant on orbits of the Hamiltonian system

$$\left\{ \begin{array}{l} \dot{x} = -\partial_{\xi'} b(x, \xi'); \\ \dot{\xi} = \partial_{x'} b(x, \xi'), \end{array} \right. \tag{2.15}$$

where $(\dot{\ }) := \partial_{x_1}$ and $b(x, \xi') := r^{\frac{1}{2}}(x, \xi')$. Let $\chi_{x_1}(y, \eta)$ be a solution to (2.15) for initial conditions close to (x_0', ξ_0') valid for $0 \leq x_1 \leq T$, say. If $T > 0$ is sufficiently small, then χ_{x_1} is invertible and $Q_0(x, \xi') = q_0(\chi_{x_1}^{-1}(x', \xi'))$ is the solution to (2.14). Now if we select q_0 compactly supported and $q_0 = 1$ in a neighbourhood of (x_0', ξ_0'), we satisfy conditions (i)-(ii) with

$$[Q_0(x, hD'), hD_1 - A_+(x, hD')] = R_1(x, hD'), \ \ \sigma_h R_1 = \mathcal{O}(h^2),$$

since ξ_1 only appears as a monomial of first order in the principal symbol. Now suppose we have $Q(x, hD')$ satisfying (i)-(ii) and

$$[Q(x, hD'), hD_1 - A_+(x, hD')] = R_j(x, hD'), \ \ \sigma_h R_j = \mathcal{O}(h^{j+1}).$$

We solve the inhomogeneous Cauchy problem

$$\left\{ \begin{array}{l} ih\left(\partial_{x_1} - H_b\right) Q_j(x, \xi') = -\sigma_h R_j(x, \xi'), \ \ x \in T^*X \\ Q_j = q_j, \ \ x_1 = 0 \end{array} \right.$$

for $Q_j = \mathcal{O}(h^j)$ and $q_j = \mathcal{O}(h^j)$, which we do by setting

$$Q_j(x_1, \chi_{x_1}(y, \eta)) = q_j(y, \eta) + (ih)^{-1} \int_0^{x_1} \sigma_h R_j(s, \chi_s(y, \eta)) ds.$$

Then $\tilde{Q} = Q + Q_j$ satisfies (i)-(ii) and

$$[\tilde{Q}(x, hD'), hD_1 - A_+(x, hD')] = R_{j+1}(x, hD'), \ \ \sigma_h R_{j+1} = \mathcal{O}(h^{j+2}).$$

By induction, the argument is finished, by setting $q_j = 0$ for $j > 0$. $\square$

Now suppose $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^{n-1})$, where we identify Y with $\mathbb{R}^{n-1}$ near (x_0'). Suppose further that $(x_0', \xi_0') \in T^*(\mathbb{R}^{n-1}) \setminus \mathrm{WF}_h \varphi$ and $\gamma_+ \cap (x_0', \xi_0') \neq \emptyset$ as before. Choose Q as in Lemma 2.6 so that

$$[Q(x, hD'), hD_1 - A_+(x, hD')] = 0$$

microlocally and if q is the principal symbol of Q, then $q(0, x_0', \xi_0') \neq 0$, but $q(0, \cdot, \cdot)$ vanishes outside a small neighbourhood of (x_0', ξ_0'). Thus $Q(x, hD')\varphi = 0$ microlocally. Suppose u_+ solves (2.13) with φ replacing $u_-|_Y$. Then Qu_+ satisfies

$$\left\{ \begin{array}{l} (hD_1 - A_+(x, hD'))\, Q(x, hD')u_+ = 0, \ \ x_1 > 0 \\ Q(x, hD')u_+ = 0, \ \ x_1 = 0 \end{array} \right.$$

microlocally. Hence by the energy estimate (2.10) $Q(x, hD')u_+ = 0$ microlocally. We conclude $\mathrm{WF}_h u_+ \subset \operatorname{char} q$.

We have proved the following proposition.

Proposition 2.7. *With the notation as in the preceding paragraphs,* $\mathrm{WF}_h u_+ \subset \chi_{x_1}(\mathrm{WF}_h \varphi)$ *and* $\mathrm{WF}_h u_- \subset \chi_{-x_1}(\mathrm{WF}_h \varphi)$*. Further, as the* x_1 *direction is reversible, if at some* $0 < x_1 < T_0$*,* $(y', \eta') \in \mathrm{WF}_h u_+(x_1, \cdot)$*, then* $\chi_{-x_1}(y', \eta') \in \mathrm{WF}_h \varphi$ *and* $\chi_{-x_1-y}(y', \eta') \in \mathrm{WF}_h u_-(y, \cdot)$ *for* $0 < y < T_0$*.*

In the special case $Y = \partial X$ we have the following lemma to connect the notions of h-wavefront set near γ_+ and γ_-.

Lemma 2.8. *Let* $(x_0', \xi_0') \in H \cap \gamma$ *be the reflection point at the boundary, let* χ_{x_1} *be a solution to* (2.15) *as above, and let* $\varphi_t = \exp(tH_p)$*. Then there is an odd diffeomorphism* $t = t(x_1)$ *and a function* $\xi_1 = \xi_1(x_1)$ *such that* $(x_1, \xi_1; \chi_{x_1})$ *lies on* γ_+ *and* $(x_1, \xi_1; \chi_{-x_1})$ *lies on* γ_- *for* $x_1 > 0$ *sufficiently small. That is,* $\chi_{x_1}(x_0', \xi_0')$ *coincides with the* (x', ξ') *components of*

$$\varphi_t(0, x_0', r^{\frac{1}{2}}(0, x_0', \xi_0'), \xi_0'),$$

Proof. Write $b(x, \xi') = r^{\frac{1}{2}}(x, \xi')$ as in the proof of Lemma 2.6, and note χ_{-x_1} is the solution to (2.15) with b replaced with $-b$. φ_t satisfies the following differential equation on γ_+:

$$\begin{cases} \partial_t x_1 = 2b \\ \partial_t x' = -2bb_{\xi'} \\ \partial_t \xi_1 = 2bb_{x_1} \\ \partial_t \xi' = 2bb_{x'}; \\ x_1(0) = 0 \\ x'(0) = x_0' \\ \xi_1(0) = b(0, x_0', \xi_0') \\ \xi'(0) = \xi_0'. \end{cases}$$

Set $(y'(x_1), \eta'(x_1)) = \chi_{x_1}(x_0', \xi_0')$,

$$t(x_1) := \int_0^{x_1} (2b(y, y'(y), \eta'(y)))^{-1} dy, \tag{2.16}$$

and calculate

$$\begin{aligned} \frac{\partial}{\partial x_1} x' &= \frac{\partial}{\partial t} x' \frac{\partial t}{\partial x_1} \\ &= -b_{\xi'}(x_1, x', \xi') \\ &= \frac{\partial}{\partial x_1} y'. \end{aligned}$$

As x' and y' have the same initial conditions, we conclude they are equal for sufficiently small x_1. For negative t, we define $t = t(x_1)$ in the incoming bicharacteristic to be the negative of that on the outgoing bicharacteristic, and a similar proof applies to χ_{-x_1}. □

Remark 2.9. With the addition of Lemma 2.8 we could write Proposition 2.7 in an equivalent form using $\exp(tH_p)$ in place of χ_{x_1}. The important thing is that the wavefront set does not depend on ξ_1.

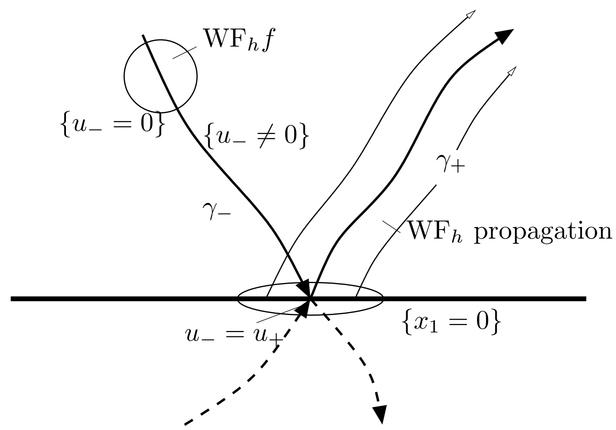

FIGURE 4. Proposition 2.7

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2.3. **Microlocal Propagator at the Boundary.** We now return to the special case where $Y = \partial X$. In the next section we will construct the Quantum Monodromy operator using the microlocal propagator. Suppose first X is a manifold without boundary. We define the forward and backward microlocal propagators of $P - z$, $I^z_\pm(t) = \exp(\mp it(P-z)/h)$, by the following evolution equation:

$$\begin{cases} hD_t I^z_\pm(t) \pm (P-z) I^z_\pm(t) = 0 \\ I^z_\pm(0) = \mathrm{id}_{L^2 \to L^2}. \end{cases}$$

In the case of a manifold without boundary, this is a well-defined semigroup satisfying $[I^z_\pm(t), P - z] = 0$ and

$$\mathrm{WF}_h I^z_\pm(t) u \subset \exp(\pm t H_p)(\mathrm{WF}_h u).$$

We will show for $P \in \mathrm{Diff}^{2,0}_h$ with homogeneous principal symbol on a manifold with boundary, the microlocal propagator can be extended in a meaningful fashion as a family of microlocally defined h-FIOs with symbols which depend discontinuously on t at points of reflection with the boundary, but still carry the commutator and wavefront set properties above.

Suppose γ reflects off ∂X at the points

$$m_\pm := (0, x_0', \pm r^{\frac{1}{2}}(0, x_0', \xi_0'), \xi_0'),$$

with the incoming and outgoing rays, $\gamma_\mp$, intersecting ∂X at $m_\mp$ respectively. Since p is assumed smooth up to the boundary, we may extend p and γ_- to a neighbourhood of m_- in $\{x_1 \leq 0\}$. We will show that functions $v(x')$ defined on ∂X can be identified with the microlocal kernel of $P - z$ in a neighbourhood of m_-. We factorize $P - z$ as in Lemma 2.1, $P - z = (hD_1 - \tilde{A}_+)(hD_1 - \tilde{A}_-)$ microlocally near m_-. Near γ_- the operator $(hD_1 - \tilde{A}_+)$ is elliptic. Thus we want to be able to solve

$$\begin{cases} (hD_1 - \tilde{A}_-) u = 0, \;\; x_1 > 0, \\ u(0, x') = v(x'), \;\; x_1 = 0 \end{cases} \tag{2.17}$$

for any boundary condition v, microlocally near m_-. The proof of the following standard Proposition can be found in [EvZw, Theorem 10.9].

Proposition 2.10. *There is a microlocal solution to* (2.17) *given by the oscillatory integral*

$$u(x) = \frac{1}{(2\pi h)^{n-1}} \int e^{i/h(\varphi(x,\xi') - \langle y', \xi' \rangle)} b(x,\xi') v(y') d\xi' dy', \tag{2.18}$$

where $b(0, x', \xi') = 1$ microlocally near (x_0', ξ_0') and φ solves the eikonal equation

$$\begin{cases} \partial_{x_1} \varphi(x,\xi') - a(x, \partial_{x'} \varphi(x,\xi')) = 0, & x_1 > 0, \\ \varphi(0, x', \xi') = \langle x', \xi' \rangle, & x_1 = 0, \end{cases} \tag{2.19}$$

with $a = \sigma_h(\tilde{A}_-)$. Further, $u(x)$ is unique microlocally.

Proposition 2.11. *Let X be a manifold with boundary, $P \in \mathit{Diff}^{2,0}_{h,db}$ be a differential operator with homogeneous principal symbol p, and assume ∂X is noncharacteristic with respect to p. Let $U_\pm \subset T^*X$ be a neighbourhood of $m_\pm \in T^*X$, and assume $P - z$ and $p - z$ are factorized near the boundary as in Lemma 2.1 and equation* (2.8) *respectively.*

(i) For each $m_0 \in \gamma_- \cap U_-$ sufficiently close to m_-, and $z \in [-\epsilon_0, \epsilon_0]$ for $\epsilon_0 > 0$ sufficiently small there exist h-FIOs, $I^z_\pm(t)$, defined microlocally near

$$\exp(\pm t H_p)(m_0) \times m_0$$

satisfying

$$\begin{cases} hD_t I^z_\pm(t) \pm (P - z)(t) I^z_\pm(t) = 0 \\ I^z_\pm(0) = \text{ id }_{L^2 \to L^2}, \end{cases} \tag{2.20}$$

for $t \neq t_1$, where $m_- = \exp(t_1 H_p)(m_0)$.

(ii) We have $[(P - z)(t), I^z_\pm(t)] = 0$ for all $t \neq t_1$ sufficiently small, and if $u(x) \in L^2$ is a microlocal solution to

$$\begin{cases} (P - z)u = f \in L^2, & x \in \mathring{X}, \\ u = 0, & x \in \partial X \end{cases} \tag{2.21}$$

near m_0, then $I^z_\pm(t)u(x)$ is a microlocal solution to (2.21) *near* $\exp(\pm t H_p)(m_0)$.

(iii) If $\mathrm{WF}_h u \subset K$, where K is a compact neighbourhood of a point m_0,

$$\mathrm{WF}_h I^z_\pm(t) u \subset \exp(\pm t H_p)(K).$$

Proof. Fix m_0. According to [EvZw, Theorem 10.18], $P - z$ may be conjugated to hD_{x_1} in a neighbourhood of m_0. Then we use the proof of Proposition 2.4 to find a solution $u_{-,1}$ to $(P - z)u = f$ near m_0. Use the microlocal forward propagator defined for a neighbourhood of γ_- extended to a neighbourhood of m_- to define $u_{-,1}$ along γ_-. That is, $I^z_+(t)u_{-,1}$ satisfies

$$(P - z) I^z_+(t) u_{-,1} = f$$

microlocally near $\exp(tH_p)(m_0)$, $0 \leq t \leq t_1$. Let $v_-(x') = I(t_1)u_{-,1}|_{\partial X}$, and use Proposition 2.10 to find a function $u_{-,2}$ satisfying

$$\begin{cases} (P - z)u_{-,2} = 0 \\ u_{-,2}|_{\partial X} = v_-(x') \end{cases}$$

microlocally near m_-. Let

$$u_- = u_{-,1} - I^z_+(-t_1) u_{-,2},$$

so that $I^z_+(t)u_-$ satisfies (2.21) microlocally near $\exp(tH_p)(m_0)$, $0 \leq t < t_1$.

Fix $m_2 = \exp(t_2 H_p)(m_0) \in \gamma_+$ sufficiently close to m_+ that we can similarly construct $I_+^z(t_2 - t)u_+$ satisfying (2.21) microlocally near $\exp(-tH_p)(m_2)$ for $0 \leq t < t_2 - t_1$. We extend $I_+^z(t)$ to be discontinuous at t_1, so that if u solves (2.21) microlocally near m_0,

$$I_+^z(t_1)u = u_- + u_+$$

with

$$\mathrm{WF}_h u_\pm \subset U_\pm \cap \{x_1 \geq 0\}.$$

We need to verify this extension of $I_+^z(t)$ satisfies (i), (ii), and (iii). For $0 \leq t < t_1$ this is clear because $I_+^z(t)$ is the usual semigroup. At t_1, we have

$$\begin{aligned}(P-z)I_+^z(t_1)u &= (P-z)(I_+^z(t_1)u_- + I_+^z(t_2 - t_1)u_+) \\ &= f_- + f_+,\end{aligned}$$

with $f_\pm = f$ microlocally near $m_\pm$. Thus (ii) and (iii) are clear.

For (i), let $A = \mathrm{Op}_h^w(a)$ be a symbol defined microlocally in a neighbourhood of $\exp(tH_p)(m_0)$. Assume $m_\pm \notin \mathrm{WF}_h A$, and let $B = \mathrm{Op}_h^w(\exp(tH_p)^* a)$. Then

$$AI_+^z(t)u = I_+^z(t)Bu$$

microlocally, and by [EvZw, Theorem 10.7], $I^z(t)$ satisfies 2.20.

The proof for $I_-^z(t)$ is similar. □

Corollary 2.12. *Let X, P, and p be as in Proposition 2.11. Suppose $\gamma(t)$ is a periodic orbit for $\exp(tH_p)$ of period T which has a finite number of transversal reflections off ∂X. Then for any $m \in \gamma(t)$, $m \cap \partial X = \emptyset$, there exist h-FIOs, $I_\pm^z(t)$, defined microlocally near $\exp(tH_p)(m) \times m$ for $0 \leq |t| \leq T$ satisfying*

(i)

$$\begin{cases} hD_t I_\pm^z(t) \pm (P-z)I_\pm^z(t) = 0 \\ I_\pm^z(0) = \mathrm{id}_{L^2 \to L^2} \end{cases}$$

for almost every t.

(ii) $[P - z, I_\pm^z(t)] = 0$, and if $u(x) \in L^2$ satisfies $(P-z)u = f \in L^2$ microlocally near m, then $I_\pm^z(t)u(x)$ satisfies

$$(P-z)I_\pm^z(t)u(x) = f(x)$$

microlocally near $\exp(\pm tH_p)(m)$.

(iii) If $\mathrm{WF}_h u \subset K$, where K is a compact neighbourhood of a point m,

$$\mathrm{WF}_h I_\pm^z(t)u \subset \exp(\pm tH_p)(K).$$

Proof. This follows immediately from Proposition 2.11 and uniqueness of solutions to ordinary differential equations. □

3. Quantum Monodromy Construction

In this section, we construct the *Quantum Monodromy* operator

$$M(z) : L^2(\mathbb{R}^{n-1}) \to L^2(\mathbb{R}^{n-1})$$

and prove some basic properties. Here we follow [SjZw1] and the somewhat simplified presentation in [SjZw2]. It is classical (see, for example, [AbMa, Theorem 28.5]) that the assumtions on p imply there exists $\epsilon_0 > 0$ such that for $-2\epsilon_0 \leq E \leq 2\epsilon_0$ there is a closed semi-hyperbolic orbit in the level set $\{p = E\}$.

Let $z \in [-\epsilon_0, \epsilon_0] \subset \mathbb{R}$. Then $p - z$ is the principal symbol of $P - z$, and $p - z$ admits a closed semi-hyperbolic orbit in the level set $\{p - z = 0\}$, say, $\gamma(z)$ of period $T(z)$.

We work microlocally in a neighbourhood of

$$\Gamma := \bigcup_{-\epsilon_0 \leq z \leq \epsilon_0} \gamma(z) \subset T^*X.$$

Fix $m_0(z) \in \gamma(z)$, $m_0(z) \cap \partial X = \emptyset$, depending smoothly on z, and set $m_1(z) = \exp(\frac{1}{2}T(z)H_p)(m_0(z))$. By perturbing $m_0(z)$ and shrinking $\epsilon_0 > 0$ if necessary, we may assume $m_1(z) \cap \partial X = \emptyset$ as well. Assume we are working with a fixed z. Define

$$\begin{aligned} \ker_{m_0(z)}(P - z) &= \{u \in L^2(\mathrm{neigh}\,(m_0(z))) : (P - z)u = 0 \\ &\qquad \text{microlocally near } m_0(z)\}, \end{aligned}$$

where "neigh $(m_0(z))$" refers to a germ, or a small arbitrary neighbourhood of $m_0(z)$ which is allowed to change from line to line. Similarly we have

$$\begin{aligned} \ker_{m_1(z)}(P - z) &= \{u \in L^2(\mathrm{neigh}\,(m_1(z))) : (P - z)u = 0 \\ &\qquad \text{microlocally near } m_1(z)\}. \end{aligned}$$

We define the forward and backward microlocal propagators $I_{\pm}^z$ as in Corollary 2.12. Then

$$I_+^z(t) : \ker_{m_0(z)}(P - z) \to \ker_{\exp(tH_p)(m_0(z))}(P - z),$$

and since

$$\exp(T(z)H_p)(m_0(z)) = m_0(z)$$

we define the *Absolute Quantum Monodromy* operator

$$\mathcal{M}(z) : \ker_{m_0(z)}(P - z) \to \ker_{m_0(z)}(P - z)$$

by

$$\mathcal{M}(z) := I_+(T(z)). \tag{3.1}$$

It is convenient to introduce an inner product structure on $\ker_{m_0(z)}(P - z)$ (see [HeSj]). For this, let $\chi \in \mathcal{C}^\infty(T^*X)$ be a microlocal cutoff supported near $\gamma(z)$ satisfying the following properties (see Figure 6):

$$\chi \equiv 1 \text{ on } \exp(tH_p)(m_0(z)) \text{ for } 0 \leq t \leq \frac{1}{2}T(0) \tag{3.2}$$

$$\chi \equiv 0 \text{ on } \exp(tH_p)(m_0(z)) \text{ for } \frac{1}{2}T(0) + \delta \leq t \leq T(0) - \delta,\ \delta > 0. \tag{3.3}$$

Let $[P, \chi]_+$ denote the part of the commutator supported near $m_0(z)$ where we use χ to denote both the function and the quantization whenever unambiguous, and for $u, v \in \ker_{m_0(z)}(P - z)$, define the *Quantum Flux product* as

$$\langle u, v \rangle_{QF} := \left\langle \frac{i}{h}[P, \chi]_+ u, v \right\rangle_{L^2(\mathrm{neigh}\,(m_0(z)))}.$$

According to [EvZw, Theorem 10.18], there is a neighbourhood of $m_0(z)$ and an h-Fourier integral operator F defined microlocally near $m_0(0)$ such that $F(P - z)F^{-1} = hD_{x_1}$ on $L^2(\widetilde{V})$, where $\widetilde{V} \subset \mathbb{R}^n$ is an open neighbourhood of $0 \in \mathbb{R}^n$. Then $\ker_{m_0(z)}(P - z)$ can be identified with $L^2(V)$, where $V \subset \mathbb{R}^{n-1}$ is an open neighbourhood of $0 \in \mathbb{R}^{n-1}$. Let

$$K(z) : L^2(V) \longleftrightarrow \ker_{m_0(z)}(P - z)$$

be the identification, and define the adjoint $K(z)^*$ with respect to the L^2 inner product on $\ker_{m_0(z)}(P-z)$. Note

$$K(z)^* : \ker_{m_0(z)}(P-z) \longleftrightarrow L^2(V)$$

is an identification as well. The following two lemmas are from [SjZw1].

Lemma 3.1. *The operator*

$$U := K(z)^* \frac{i}{h}[P,\chi]_+ K(z) : L^2(V) \to L^2(V)$$

is positive definite. Setting $\widetilde{K}(z) = K(z)U^{\frac{1}{2}}$, we have

$$\widetilde{K}(z)^* \frac{i}{h}[P,\chi]_+ \widetilde{K}(z) = \text{ id } : L^2(V) \to L^2(V).$$

Proof. Using [EvZw, Theorem 10.18], we write

$$\begin{aligned} \left\langle K(z)^* \frac{i}{h}[P,\chi]_+ K(z)v, v \right\rangle_{L^2(V)} &= \langle \partial_{x_1}\chi K(z)v, K(z)v\rangle_{L^2(\text{neigh}\,(m_0(z)))} \\ &\geq C^{-1}\|v\|^2. \end{aligned}$$

□

Remark 3.2. In light of Lemma 3.1, we replace $K(z)$ with $\widetilde{K}(z)$ and write

$$K(z)^{-1} = K(z)^* \frac{i}{h}[P,\chi]_+.$$

Lemma 3.3. *The Quantum Flux product $\langle \cdot,\cdot\rangle_{QF}$ does not depend on the choice of χ satisfying (3.2-3.3). In addition, $\mathcal{M}(z)$ is unitary on $\ker_{m_0(z)}(P-z)$ with respect to this product.*

Proof. Suppose $u, v \in L^2(V)$ and suppose $\widetilde{\chi}$ is another function satisfying (3.2-3.3) which agrees with χ near $m_1(z)$. Then $[P,\widetilde{\chi}-\chi]_+ = [P,\widetilde{\chi}-\chi]$, $(P-z)K(z)u = 0$, and $K(z)^*(P-z) = ((P-z)K(z))^*$ imply

$$\left\langle \frac{i}{h}[P,\widetilde{\chi}-\chi]_+ K(z)u, K(z)v \right\rangle = \left\langle \frac{i}{h}(\widetilde{\chi}-\chi)K(z)u, (P-z)K(z)v \right\rangle = 0.$$

To see $\mathcal{M}(z)$ is unitary, observe for $\tilde{u} \in \ker_{m_0(z)}(P-z)$,

$$\begin{aligned} &\left\langle \frac{i}{h}[P,\chi]_+ I_+^z(T(z))\tilde{u}, I_+^z(T(z))\tilde{u} \right\rangle = \\ &= \left\langle \frac{i}{h}\left[P, I_-^z(T(z))\chi I_+^z(T(z))\right]_+ \tilde{u}, \tilde{u} \right\rangle \\ &= \left\langle \frac{i}{h}\left[P,\widetilde{\chi}\right]_+ \tilde{u}, \tilde{u} \right\rangle, \end{aligned}$$

where $\widetilde{\chi} = \exp(TH_p)^*\chi$ satisfies (3.2-3.3). □

Next we restrict our attention to $L^2(V)$ by defining the *Quantum Monodromy operator* $M(z) : L^2(V) \to L^2(V)$ by

$$M(z) = K(z)^{-1}\mathcal{M}(z)K(z).$$

Lemma 3.4. *$M(z) : L^2(V) \to L^2(V)$ is unitary, and $M(z)$ is a (microlocally unitary) quantization of the Poincaré map S.*

Proof. Let $u \in L^2(V)$. We calculate:

$$\begin{aligned}
& \langle M(z)u, M(z)u\rangle_{L^2(V)} = \\
&= \left\langle K(z)^{-1}\mathcal{M}(z)K(z)u, K(z)^{-1}\mathcal{M}(z)K(z)u\right\rangle_{L^2(V)} \\
&= \left\langle (K(z)^*)^{-1}K(z)^{-1}\mathcal{M}(z)K(z)u, \mathcal{M}(z)K(z)u\right\rangle_{L^2(\text{neigh}\,(m_0(z)))} \\
&= \left\langle \frac{i}{h}[P,\chi]_+\mathcal{M}(z)K(z)u, \mathcal{M}(z)K(z)u\right\rangle_{L^2(\text{neigh}\,(m_0(z)))} \\
&= \langle K(z)u, K(z)u\rangle_{L^2(\text{neigh}\,(m_0(z)))} \\
&= \langle u, u\rangle_{L^2(V)}\,.
\end{aligned}$$

In order to prove $M(z)$ is the quantization of the Poincaré map, we will use [EvZw, Theorem 10.7]. We need to prove for pseudodifferential operators $A, B \in \psi_h^{0,0}(V)$ such that $\sigma_h(B) = S^*\sigma_h(A)$, we have $AM(z) = M(z)B$. Without loss of generality, we write $x' = (x_2, \ldots, x_n) \in V$ for the variables in V and $x = (x_1, x') \in \text{neigh}\,(\gamma(z))$ for the variables in X near γ. Then for $v \in L^2(V) \cap \mathcal{C}^\infty(V)$

$$\begin{aligned}
& M(z)B(x', hD_{x'})v(x') = \\
&= K(z)^{-1}\mathcal{M}(z)K(z)B(x', hD_{x'})v(x') \\
&= K(z)^{-1}I_+^z(T(z))B(x', hD_{x'})I_-^z(T(z))I_+^z(T(z))K(z)v(x') \\
&= K(z)^{-1}\mathrm{Op}\left((\exp(TH_p))^*\,\sigma_h(B)\right)(x, hD_x)I_+^z(T(z))K(z)v(x') \\
&= A(x', hD_{x'})M(z)v(x').
\end{aligned}$$

□

4. The Grushin Problem

4.1. **Motivation of the Grushin Problem.** In this section we follow [SjZw1] and show how the Quantum Monodromy operator arises naturally in the context of a *Grushin Problem* near γ. This is a generalization of the linear algebra Grushin problem: Suppose

$$\begin{aligned}
A &: H \to H, \\
B &: H_- \to H, \\
C &: H \to H_+, \text{ and} \\
D &: H_- \to H_+
\end{aligned}$$

are matrices acting on finite dimensional Hilbert spaces H, H_-, and H_+, and

$$\begin{pmatrix} \alpha & \beta \\ \sigma & \delta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1},$$

where

$$\begin{aligned}
\alpha &: H \to H, \\
\beta &: H_+ \to H, \\
\sigma &: H \to H_-, \text{ and} \\
\delta &: H_+ \to H_-.
\end{aligned}$$

Then A is invertible if and only if δ is invertible, in which case

$$A^{-1} = \alpha - \beta\delta^{-1}\sigma.$$

It appears counterintuitive at first that understanding the invertibility of a larger matrix might be easier than understanding the invertibility of a submatrix. However, when the entries are operators instead of matrices, the situation may change. In the next section we will see that introducing a matrix of operators will allow us to understand microlocal invertibility of $P-z$ near a periodic orbit by constructing a parametrix in the *double cover* of a neighbourhood of the orbit.

4.2. **The Grushin Problem Reduction.** Throughout this section, we suppress the dependence on z whenever unambiguous for ease in exposition. We will build operators $R_+ = R_+(z) : \mathcal{D}'(X) \to \mathcal{D}'(V)$ and $R_- = R_-(z) : \mathcal{D}'(V) \to \mathcal{D}'(X)$ such that

$$\mathcal{P} := \begin{pmatrix} \frac{i}{h}(P-z) & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}'(X) \times \mathcal{D}'(V) \to \mathcal{D}'(X) \times \mathcal{D}'(V)$$

has microlocal inverse

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \tag{4.1}$$

near $\gamma \times (0,0)$, where E, E_+, and E_- will be defined later, and

$$E_{-+} = I - M(z).$$

The following construction of the solution to the Grushin problem is from [SjZw1], with the addition here that we allow $\gamma(z)$ to reflect transversally off the boundary of ∂X. Recall $\chi \in \mathcal{C}_c^\infty(T^*X)$ satisfies (3.2-3.3), and begin by setting

$$R_+ = K^* \frac{i}{h} [P, \chi]_+ .$$

Then if u satisfies $(P-z)u = 0$ microlocally near $m_0(z)$, $R_+ u$ is the microlocal Cauchy data. That is, for $v \in L^2(V)$, $u = Kv$ is a solution to the microlocal Cauchy problem

$$\begin{cases} (P-z)u = 0, \\ R_+ u = v \end{cases} \tag{4.2}$$

near $\gamma \times (0,0)$. To construct a global solution, let $K_f(t) := I_+(t)K$ and $K_b(t) := I_-(t)K$ be the forward and backward (respectively) Cauchy problem solution operators. Note for $t \sim T/2$ we have

$$\begin{aligned} K_f(t) &= I_+(t)K \\ &= I_-(t)KK^{-1}\mathcal{M}(z)K \\ &= K_b(t)M(z), \end{aligned} \tag{4.3}$$

so microlocally near $m_1 \times (0,0)$ we have $K_f = K_b M(z)$. Now for Ω a neighbourhood of γ, we can solve (4.2) in $\Omega \setminus \operatorname{neigh}(m_1)$. To do this, set

$$E_+ v = \chi K_f v + (1-\chi) K_b v, \tag{4.4}$$

so $E_+ v$ satisfies

i) $E_+ v = Kv$ in a neighbourhood of $m_0(z)$

ii) $R_+ E_+ = \mathrm{id}$ microlocally near $(0,0) \times (0,0) \in (T^*V)^2$.

With $[\cdot,\cdot]_-$ denoting the part of the commutator supported near $m_1(z)$, we calculate:

$$(P-z)E_+ v = [P,\chi]_- K_f v - [P,\chi]_- K_b v,$$

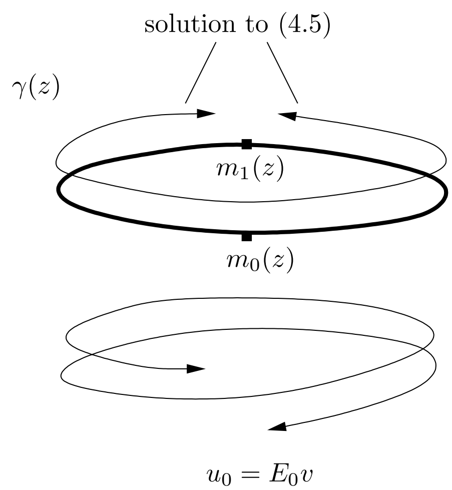

FIGURE 5. Microlocal solution to (4.5) and construction of global solution to (4.6).

since $K_f = K_b$ microlocally near $m_0(z) \times (0,0)$ and $(P-z)I_\pm(t)Kv = 0$ microlocally near $\exp(tH_p)(m_0(z)) \times (0,0)$. According to (4.3), we can then write

$$(P-z)E_+v = [P,\chi]_- K_b(M(z) - \mathrm{id}\,)v.$$

For $v \in L^2(V)$, we set

$$\begin{aligned} u &= E_+ v, \\ u_- &= E_{-+} v, \text{ and} \\ R_- &= \frac{i}{h}[P,\chi]_- K_b. \end{aligned}$$

We have solved the following problem microlocally in $(\Omega \setminus \mathrm{neigh}\,(m_1(z)))^2$ (see Figure 5):

$$\left\{ \begin{array}{r} \frac{i}{h}(P-z)u + R_- u_- = 0 \\ R_+ u = v \end{array} \right. . \tag{4.5}$$

Thus if $\mathcal{P}^{-1}$ exists, it is necessarily given by (4.1), where E and E_- have yet to be defined.

For $\epsilon > 0$ let

$$(\Omega \times_\epsilon \Omega)_\pm := \left\{ \left(\bigcup_{m \in \Omega} (\exp \pm tH_p)m, m \right) \bigcap \Omega \times \Omega : -\epsilon < t < T - 2\epsilon \right\}.$$

We will define L_f and L_b, the forward and backward fundamental solutions (respectively) of $i(P-z)/h$, which will be defined microlocally on $(\Omega \times_\epsilon \Omega)_\pm$ respectively. By [EvZw, Theorem 10.18], we can conjugate $i(P-z)/h$ to ∂_{x_1} microlocally near the point $m_0(z)^2 \in (T^*X)^2$. Then the local fundamental solutions L_f^0 and L_b^0 are

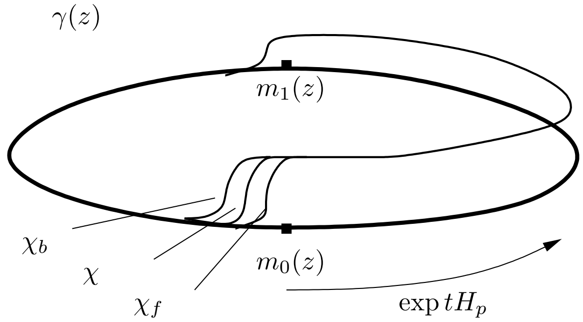

FIGURE 6. The cutoffs χ_b, χ, and χ_f.

given by

$$\begin{aligned} L_f^0 v(x) &= \int_{-\infty}^{x_1} v(y,x')dy, \text{ and} \\ L_b^0 v(x) &= -\int_{x_1}^{\infty} v(y,x')dy, \end{aligned}$$

while L_f and L_b are now given microlocally near $\exp(\pm tH_p)m_0(z) \times m_0(z)$, respectively, by

$$\begin{aligned} L_f &= I_+^z(t)L_f^0 \text{ and} \\ L_b &= I_-^z(t)L_b^0. \end{aligned}$$

It is convenient to introduce two new microlocal cutoffs χ_f and χ_b satisfying (3.2-3.3) and in addition,

$$\begin{aligned} &\chi \equiv 1 \text{ on } \operatorname{supp} \chi_f \cap W_+, \\ &\chi_b \equiv 1 \text{ on } \operatorname{supp} \chi \cap W_+, \end{aligned}$$

where W_+ is a neighbourhood of $m_0(z)$ containing the support of $[P,\chi]_+$ (see Figure 6). For $v \in L^2(\Omega)$, set

$$\tilde{u} = L_f(I-\chi)v,$$

and observe $(P-z)\tilde{u} = 0$ past the support of $(I-\chi)$ in the direction of the H_p flow. In particular, $(P-z)\tilde{u} = 0$ on $\operatorname{supp} \chi_f$. Then past $\operatorname{supp}(I-\chi)$ in the direction of the H_p flow,

$$\begin{aligned} \tilde{u} &= KK^* \frac{i}{h} [P,\chi_f]_+ \tilde{u} \\ &= KK^* \frac{i}{h} [P,\chi_f]_+ L_f(I-\chi)v. \end{aligned}$$

Let $\widetilde{I}_+(t)$ be the extension of $I_+(t)$ to $T \le t \le 2T-\epsilon$, and let $\widetilde{K}_f = \widetilde{I}_+K$. Let $\widetilde{\Omega}$ denote the double covering space of Ω. Then in $\widetilde{\Omega}$,

$$\tilde{u} = \widetilde{K}_f K^* \frac{i}{h} [P,\chi_f]_+ L_f(I-\chi)v.$$

We define $\hat{u} = L_b\chi v$ and $\widetilde{K}_b = \widetilde{I}_-K$ similar to $\widetilde{K}_f$ so that

$$\hat{u} = \widetilde{K}_b K^* \frac{i}{h} [P,\chi_b]_+ L_b\chi v.$$

We think of $\tilde{u}$ and $\hat{u}$ as being double-valued on Ω and write $L_{ff}v$ and $L_{bb}v$ to denote the second branches respectively in a neighbourhood of $m_1(z)$. Let W_- denote a neighbourhood of $m_1(z)$, and define (see Figure 5)

$$u_0 = E_0 v := \begin{cases} L_b\chi v + L_f(I-\chi)v & \text{outside } W_-, \\ L_b\chi v + (I-\chi)L_{bb}\chi v + L_f(I-\chi)v + \chi L_{ff}(I-\chi)v & \text{in } W_- \end{cases}.$$

Now we apply $i(P-z)/h$ to E_0v in W_-:

$$\begin{aligned} \frac{i}{h}(P-z)E_0 v &= v - \frac{i}{h}\left[P,\chi\right]_- L_{bb}\chi v + \frac{i}{h}\left[P,\chi\right]_- L_{ff}(I-\chi)v \\ &= v - \frac{i}{h}\left[P,\chi\right]_- K_b K^* \frac{i}{h}\left[P,\chi\right]_+ L_b\chi v \\ &\qquad + \frac{i}{h}\left[P,\chi\right]_- K_f K^* \frac{i}{h}\left[P,\chi\right]_+ L_f(I-\chi)v \\ &= v - \frac{i}{h}\left[P,\chi\right]_- K_b\Big(K^*\frac{i}{h}\left[P,\chi\right]_+ L_b\chi v \\ &\qquad - M(z)K^*\frac{i}{h}\left[P,\chi\right]_+ L_f(I-\chi)v\Big), \end{aligned}$$

where we have used $K_f = K_b M(z)$ in W_- and dropped the tilde and hat notation when thinking of second branches. We have solved the following problem:

$$\frac{i}{h}(P-z)E_0 v + R_- E_{0,-}v = v, \tag{4.6}$$

with

$$R_- = \frac{i}{h}\left[P,\chi\right]_- K_b$$

as above, and

$$E_{0,-}v := K^*\frac{i}{h}\left[P,\chi\right]_+ L_b\chi v - M(z)K^*\frac{i}{h}\left[P,\chi\right]_+ (I-\chi)v.$$

Recalling the structure of $\mathcal{E}$ and $\mathcal{P}$, we calculate

$$\mathcal{P}\mathcal{E} = \begin{pmatrix} \frac{i}{h}(P-z)E + R_-E_- & \frac{i}{h}(P-z)E_+ + R_-E_{-+} \\ R_+E & R_+E_+ \end{pmatrix},$$

so that if $\mathcal{E}$ is to be a microlocal right inverse of $\mathcal{P}$ near $\gamma\times(0,0)$, we require

$$\frac{i}{h}(P-z)E + R_-E_- = \text{id} : L^2(\Omega)\to L^2(\Omega), \tag{4.7}$$

$$\frac{i}{h}(P-z)E_+ + R_-E_{-+} = 0 : L^2(V)\to L^2(\Omega), \tag{4.8}$$

$$R_+E = 0 : L^2(\Omega)\to L^2(V) \text{ and,} \tag{4.9}$$

$$R_+E_+ = \text{id} : L^2(V)\to L^2(V) \tag{4.10}$$

microlocally. Note (4.8) and (4.10) are satisfied according to (4.5). Owing to (4.10), if we write $E = (I - E_+R_+)\tilde{E}$ for some $\tilde{E}$, then

$$R_+E = R_+(I - E_+R_+)\tilde{E} = (I - R_+E_+)R_+\tilde{E} = 0,$$

and comparing with (4.7) we see $\tilde{E} = E_0$,

$$E = E_0 + E_+R_+E_0,$$

and

$$E_- = E_{0,-} - E_{-+}R_+E_0.$$

Thus $\mathcal{E}$ is a right inverse. To see it is also a left inverse, observe

$$\begin{aligned} R_+^* &= \frac{i}{h}\,[P,\chi]_+\,K, \text{ and} \\ R_-^* &= K_b^* \frac{i}{h}\,[P,\chi]_-\,, \end{aligned}$$

together with

$$K_b^* \frac{i}{h}\,[P,\chi]_-\,K_b = -\operatorname{id}$$

implies

$$K_b^* \frac{i}{h}\,[P,(I-\chi)]_-\,K_b = \operatorname{id}.$$

In other words, after exchanging χ with $1-\chi$, W_+ with W_-, and K with K_b, R_+^* has the same form as R_- and R_-^* has the same form as R_+. Thus

$$\mathcal{P}^* = \begin{pmatrix} \frac{i}{h}(P-z) & R_+^* \\ R_-^* & 0 \end{pmatrix}$$

has the same form as $\mathcal{P}$ and hence has a microlocal right inverse, say

$$\mathcal{F}^* := \begin{pmatrix} F & F_+ \\ F_- & F_{-+} \end{pmatrix}^*$$

Then $\mathcal{P}^*\mathcal{F}^* = \operatorname{id}$ implies $\mathcal{F}\mathcal{P} = \operatorname{id}$, so

$$\mathcal{F} = \mathcal{F}\mathcal{P}\mathcal{E} = \mathcal{E}$$

implies $\mathcal{F} = \mathcal{E}$.

As every operator used in the preceding construction depends holomorphically on $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$, we have proved the following Proposition, which is from [SjZw1]:

Proposition 4.1. *With $\mathcal{P}$ and $\mathcal{E}$ as above and $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$, $\mathcal{E}$ is a microlocal inverse for*

$$\mathcal{P} : L^2(\Omega) \times L^2(V) \to L^2(\Omega) \times L^2(V)$$

near $\gamma \subset T^*X$, and in addition,

$$\|\mathcal{E}\|_{L^2(\Omega)\times L^2(V) \to L^2(\Omega)\times L^2(V)} \leq C.$$

4.3. **Comparing $P-z$ to $M(z)$.** As a consequence of Proposition 4.1 and motivated by the linear algebra Grushin problem, the following two theorems show quantitatively that $P-z$ is invertible if and only if $I-M(z)$ is invertible.

Theorem 4. *Let $M(z) : L^2(V) \to L^2(V)$ be the Quantum Monodromy operator, $(P-z)$ and R_+ as above. Suppose $A \in \Psi_h^{0,0}(T^*X)$ is a microlocal cutoff with wavefront set sufficiently close to $\gamma \subset T^*X$ and $B \in \Psi_h^{0,0}(T^*V)$ is a microlocal cutoff with wavefront set sufficiently close to $(0,0) \in T^*V$. Then there exists $\epsilon_0 > 0$, $c_0 > 0$, and $h_0 > 0$ such that, with $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$ and $0 < h < h_0$,*

$$\begin{aligned} &\|(P-z)u\|_{L^2(X)} \geq \\ &\geq C^{-1} h \left(\|B(I-M(z))R_+ u\|_{L^2(V)} - \|(I-A)u\|_{L^2(X)} \right) \\ &\quad -\mathcal{O}(h^\infty)\|u\|_{L^2(X)}. \end{aligned} \tag{4.11}$$

Further,

$$\begin{aligned} &\|Au\|_{L^2(X)} \leq \\ (4.12) \qquad &\leq C\left(\|R_+u\|_{L^2(V)} + h^{-1}\|(P-z)u\|_{L^2(X)} + \|(I-A)u\|_{L^2(X)}\right) \\ &\quad + \mathcal{O}(h^\infty)\|u\|_{L^2(X)}. \end{aligned}$$

Proof. That $\mathcal{E}$ is a microlocal left inverse for $\mathcal{P}$ means in particular that for A and B as in the statement of the theorem,

$$\frac{i}{h}E_-(P-z) + E_{-+}R_+ = l_+ + \mathcal{O}(h^\infty)_{L^2(X)\to L^2(V)}, \tag{4.13}$$

where

$$Bl_+A = \mathcal{O}(h^\infty)_{L^2(X)\to L^2(V)}.$$

Since (4.11) is only concerned with injectivity, we note that by replacing E_- and E_{-+} with $\tilde{E}_- = BE_-$ and $\tilde{E}_{-+} = BE_{-+}$ respectively in (4.13) doesn't change the fact that $\mathcal{E}$ is a microlocal *left* inverse. Thus

$$\frac{i}{h}\tilde{E}_-(P-z) + \tilde{E}_{-+}R_+ = \tilde{l}_+ + \mathcal{O}(h^\infty)_{L^2(X)\to L^2(V)}, \tag{4.14}$$

with

$$\tilde{l}_+A := Bl_+A = \mathcal{O}(h^\infty)_{L^2(X)\to L^2(V)},$$

and for $u \in L^2(X)$,

$$\tilde{E}_-(P-z)u + \frac{h}{i}\tilde{E}_{-+}R_+u = \frac{h}{i}\tilde{l}_+(I-A)u + \mathcal{O}(h^\infty)\|u\|_{L^2(X)},$$

hence (4.11).

For (4.12), we note $\mathcal{E}\mathcal{P} = \mathrm{id}$ microlocally gives also

$$\frac{i}{h}E(P-z) + E_+R_+ = \mathrm{id}_{L^2(X))\to L^2(X)} + l, \tag{4.15}$$

where

$$AlA = \mathcal{O}(h^\infty)_{L^2(X)\to L^2(X)}.$$

Similar to (4.14), we replace E and E_+ with $\tilde{E} = AE$ and $\tilde{E}_+ = AE_+$ without changing that $\mathcal{E}$ is a microlocal left inverse of $\mathcal{P}$, and from (4.15), we get for $u \in L^2(X)$

$$\frac{i}{h}\tilde{E}(P-z)u + \tilde{E}_+R_+u = Au + \tilde{l}u + \mathcal{O}(h^\infty)\|u\|_{L^2(X)},$$

Using $\tilde{l}Au := AlAu = \mathcal{O}(h^\infty)u$, we get

$$C\left(\|(P-z)u\|_{L^2(X)} + h\|R_+u\|_{L^2(V)}\right) \geq h\|Au\|_{L^2(X)} - h\|(I-A)u\|_{L^2(X)},$$

which is (4.12). $\square$

Using that $\mathcal{E}$ is a microlocal right inverse for $\mathcal{P}$ we obtain the following theorem, which completes the correspondence between $I - M(z)$ and $P - z$.

Theorem 5. *Suppose* $A \in \Psi^{0,0}(X)$, $B \in \Psi^{0,0}(V)$ *satisfy*

$$\begin{aligned} &A \equiv 1 \textit{ microlocally near } \gamma, \\ &A \equiv 0 \textit{ microlocally away from } \gamma, \\ &B \equiv 1 \textit{ microlocally near } (0,0), \\ &B \equiv 0 \textit{ microlocally away from } (0,0). \end{aligned}$$

Suppose $u \in L^2(X)$ *satisfies*

$$Au = u + \mathcal{O}(h^\infty)\|u\|_{L^2(X)},$$

and $v \in L^2(V)$ *satisfies*

$$Bv = v + \mathcal{O}(h^\infty)\|v\|_{L^2(V)}.$$

Then we have

$$A\frac{i}{h}(P-z)Eu + AR_-E_-u = u + \mathcal{O}(h^\infty)\|u\|_{L^2(X)}, \quad \textit{and} \tag{4.16}$$

$$A\frac{i}{h}(P-z)E_+v + AR_-(I - M(z))v = \mathcal{O}(h^\infty)\|v\|_{L^2(V)}. \tag{4.17}$$

Remark 4.2. The utility of (4.17) is that in §9, where γ will be assumed elliptic instead of semi-hyperbolic, we construct $v \in L^2(V)$ concentrated near $(0,0)$ satisfying

$$(I - M(z))v = \mathcal{O}(h^N), \quad \forall N$$

then $u := E_+v$ satisfies

$$(P-z)u = \mathcal{O}(h^{N+1})\|u\|_{L^2(X)}$$

microlocally near γ. This provides essentially a converse to our Main Theorem.

Proof of Theorem 5. From Proposition 4.1, if we multiply $\mathcal{P}$ by $\mathcal{E}$ on the right, we get

$$\frac{i}{h}(P-z)E + R_-E_- = \mathrm{id}_{L^2(X)\to L^2(X)} + r, \tag{4.18}$$

$$\begin{aligned} \frac{i}{h}(P-z)E_+ + R_-(I - M(z)) &= r_-, \\ R_+E &= r_+, \\ R_+E_+ &= \mathrm{id}_{L^2(V)\to L^2(V)} + r_{-+}, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} ArA &= \mathcal{O}(h^\infty)_{L^2(X)\to L^2(X)}, \\ Ar_-B &= \mathcal{O}(h^\infty)_{L^2(V)\to L^2(X)}, \\ Br_+A &= \mathcal{O}(h^\infty)_{L^2(X)\to {}^2(V)}, \text{ and} \\ Br_{-+}B &= \mathcal{O}(h^\infty)_{L^2(V)\to L^2(V)}. \end{aligned}$$

Hence (4.18-4.19) imply for any $u \in L^2(X)$, $v \in L^2(V)$,

$$A\frac{i}{h}(P-z)Eu + AR_-E_-u = Au + Ar(I-A)u + \mathcal{O}(h^\infty)\|u\|_{L^2(X)}$$

and

$$A\frac{i}{h}(P-z)E_+v + AR_-(I-M(z))v = Ar_-(I-B)v + \mathcal{O}(h^\infty)\|v\|_{L^2(V)},$$

which is (4.16-4.17). ☐

5. The Model Case

In this section we indicate how Theorem 4 can be used to estimate $P - z$ in the model case. Let $\dim X = 2$, and assume t parametrizes $\gamma = \gamma(0)$, and τ is the dual variable to t. Then our model for p near γ is the symbol

$$p = \tau + \lambda x \xi,$$

with $\lambda > 0$. We have

$$H_p = \partial_t + \lambda(x\partial_x - \xi\partial_\xi),$$

and the Poincaré map $S : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$S = \left(\begin{array}{cc} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{array} \right).$$

We want a deformation of the identity into S, that is a smooth family of symplectomorphisms κ_t such that $\kappa_0 = \text{ id}$ and $\kappa_1 = S$. This is clear in the model case:

$$\kappa_t = \exp t \left(\begin{array}{cc} \lambda & 0 \\ 0 & -\lambda \end{array} \right).$$

According to [EvZw, Theorem 10.1], we can find a time-dependent effective Hamiltonian $q_t = q_t(x, \xi)$ such that

$$\frac{d}{dt}\kappa_t = (\kappa_t)_* H_{q_t}.$$

In the model case, this is again clear: $q_t = \lambda x \xi$, independent of t.

We know in general if $M(z)$ is the Quantum Monodromy operator it is an h-FIO associated to the graph of S, which means our model is $M(z) = M^z(1)$ for $M^z(t)$ a family of h-FIOs satisfying

$$\left\{ \begin{array}{l} hD_t M^z(t) + Q(t)M^z(t) = 0, \\ M^z(0) = \text{ id} \end{array} \right.$$

where $Q(t) = \mathrm{Op}\,(q_t)$ for the effective Hamiltonian q_t as above. In the model case, q does not depend on t or z, so with $Q = \mathrm{Op}\,(q)$, $M^z(t)$ is just the semigroup

$$M(t) = \exp\left(-\frac{i}{h}tQ\right).$$

The basic idea is $M(t)$ is unitary, but $e^{-G^w}M(t)e^{G^w}$ is not for G with real principal symbol (if it exists). Further, in the model case, if G is independent of t,

$$e^{-G^w}M(t)e^{G^w} = \exp\left(-\frac{i}{h}te^{-G^w}Qe^{G^w}\right),$$

and it will suffice to show $e^{-G^w}Qe^{G^w}$ has an imaginary part of fixed size comparable to h.

As in [Chr1, Lemma 2.4], for $u \in L^2(\mathbb{R}^n)$ we define $T_{h,\tilde{h}}$ by

$$T_{h,\tilde{h}}u(X) := \left(h/\tilde{h}\right)^{\frac{n}{4}} u\left(\left(h/\tilde{h}\right)^{\frac{1}{2}} X\right). \tag{5.1}$$

We then conjugate $M(t)$ to $M_1(t) = T_{h,\tilde{h}} M(t) T^{-1}_{h,\tilde{h}}$, and observe $M_1(t)$ satisfies the evolution equation

$$\begin{aligned} hD_t M_1(t) &= -T_{h,\tilde{h}} Q M(t) T^{-1}_{h,\tilde{h}} && (5.2)\\ &= -T_{h,\tilde{h}} Q T^{-1}_{h,\tilde{h}} T_{h,\tilde{h}} M(t) T^{-1}_{h,\tilde{h}} && (5.3)\\ &= -Q_1 M_1(t), && (5.4)\end{aligned}$$

where

$$Q_1 = T_{h,\tilde{h}} Q T^{-1}_{h,\tilde{h}} \in \Psi^{-\infty,0,0}_{-\frac{1}{2}}$$

microlocally. We write $q_1(X,\Xi) = \sigma_{\tilde{h}}(Q_1)$, where

$$q_1(X,\Xi) = \lambda\Big(h/\tilde{h}\Big) X\Xi + \mathcal{O}(h^2 + \tilde{h}^2),$$

as in [Chr1, Lemma 2.4].

Now we define the *escape function*

$$G(X,\Xi) = \frac{1}{2}\log\left(\frac{1+X^2}{1+\Xi^2}\right),$$

and according to a result of Bony-Chemin [BoCh] (see also [Chr1, Lemma 2.1]), we can form the family of operators

$$e^{sG^w},$$

where G^w is the Weyl quantization of G in the $\tilde{h}$ calculus and $|s|$ is sufficiently small. Let

$$\widetilde{M}(t) = e^{-sG^w} M_1(t) e^{sG^w},$$

whence

$$hD_t \widetilde{M}(t) = -\widetilde{Q}\widetilde{M}(t)$$

for

$$\widetilde{Q} = e^{-sG^w} Q_1 e^{sG^w}$$

by a similar argument to (5.2-5.4). We write

$$\widetilde{Q} = \exp(-s\operatorname{ad}_{G^w}) Q_1,$$

with

$$\operatorname{ad}^k_{G^w} Q_1 = \mathcal{O}_{L^2\to L^2}\left(h\tilde{h}^{k-1}\right),$$

and

$$[Q_1, G^w] = -i\tilde{h}\operatorname{Op}^w_{\tilde{h}}(H_{q_1}G) + \mathcal{O}(h^{3/2}\tilde{h}^{3/2}).$$

We have

$$\begin{aligned} H_{q_1} G &= \lambda\Big(h/\tilde{h}\Big)\left(\frac{X^2}{1+X^2} + \frac{\Xi^2}{1+\Xi^2}\right)\\ &=: \lambda\Big(h/\tilde{h}\Big) A,\end{aligned}$$

so that

$$\widetilde{Q} = Q_1 - ish\operatorname{Op}^w_{\tilde{h}}(A) + sE_1^w + s^2E_2^w,$$

with $E_1 = \mathcal{O}(h^{3/2}\tilde{h}^{3/2})$ and $E_2 = \mathcal{O}(h\tilde{h})$. Since A is roughly the harmonic oscillator (see [Chr1, Lemma 5.1]),

$$\langle \operatorname{Op}_{\tilde{h}}^{w}(A)\, U, U\rangle \geq \frac{\tilde{h}}{C}\|U\|^2$$

independently of h, so that

$$\operatorname{Im}\left\langle \widetilde{Q}U, U\right\rangle \leq -\frac{h\tilde{h}}{C}\|U\|^2. \tag{5.5}$$

Thus with $\tilde{h} > 0$ small but fixed,

$$\widetilde{M}(1) = \exp\left(-\frac{i}{h}(\operatorname{Re}\widetilde{Q} + i\operatorname{Im}\widetilde{Q})\right)$$

and by (5.5),

$$\left\|\widetilde{M}(1)\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leq r < 1. \tag{5.6}$$

For $u \in L^2(\mathbb{R})$ and $U = T_{h,\tilde{h}}u$, we have by (5.5) and (5.6)

$$\operatorname{Re}\left\langle (I - \widetilde{M}(1))U, U\right\rangle \geq C^{-1}\|U\|^2$$

for some $0 < C < \infty$. Define the operator K^w by

$$e^{sK^w} = T_{h,\tilde{h}}^{-1} e^{sG^w} T_{h,\tilde{h}}. \tag{5.7}$$

We have shown that

$$\operatorname{Re}\left\langle e^{-sK^w}(I - M)e^{sK^w}u, u\right\rangle \geq C^{-1}\|u\|^2.$$

Since $\|\exp(\pm sK^w)\|_{L^2\to L^2} = \mathcal{O}(h^{-N})$ for some N, we have

$$\operatorname{Re}\langle (I - M)u, u\rangle \geq Ch^N\|u\|^2.$$

6. The Linearization

6.1. **Symplectic Linear Algebra and Matrix Logarithms.** In this section, we will show how to reduce the case of a general Poincaré map with a fixed point to studying the quadratic Birkhoff normal forms. We assume as in the introduction that the eigenvalues of modulus one obey the nonresonance assumption (1.2).

We begin by tackling the problem of negative real eigenvalues and eigenvalues of modulus 1 of the linearized Poincaré map. Let $S : W_1 \to W_2$, $W_1, W_2 \subset \mathbb{R}^{2n-2}$, be a local symplectic map, $S(0,0) = (0,0)$, which we have identified with its coordinate representation. As in the proof of [EvZw, Theorem 10.1], we consider the polar decomposition of $dS(0,0)$:

$$dS(0,0) = \exp(-JF)\exp(B),$$

with F and B real valued and $\exp(B)$ positive definite and symplectic. Specifically, $\exp(-JF)$ describes the action due to the eigenvalues of modulus 1 as well as the rotation inherent in the negative real eigenvalues. We consider first $A = \exp(B)$. We denote by $\{\mu_j\}$ the eigenvalues of A and by $\{\tilde{\mu}_j\}$ the eigenvalues of $dS(0,0)$. Let μ be an eigenvalue of A. Then A symplectic implies if $\mu > 1$ is real μ^{-1} is also an eigenvalue, and if μ is complex, $|\mu| > 1$, μ^{-1}, $\overline{\mu}$, and $\overline{\mu}^{-1}$ are also eigenvalues. If $\tilde{\mu}$, $|\tilde{\mu}| = 1$ is an eigenvalue of $dS(0,0)$, then $\overline{\tilde{\mu}} = \mu^{-1}$ is also an eigenvalue $dS(0,0)$, but we will see neither of these contributes to A. If $\tilde{E}_\mu$ is the generalized complex eigenspace of μ, then we can put A into complex Jordan form over $\tilde{E}_\mu$. To keep the change of variables symplectic, we observe that $\tilde{E}_{\mu^{-1}}$ is the dual eigenspace to $\tilde{E}_\mu$, so if

$$A_\mu := A|_{\tilde{E}_\mu} = \begin{pmatrix} \mu & 1 & 0 & \dots\dots \\ 0 & \mu & 1 & 0 \;\; \dots \\ \vdots & \ddots & \ddots & \dots\dots \\ 0 & \dots & 0 & \mu \;\;\; 1 \\ 0 & \dots\dots\dots\dots & & \mu \end{pmatrix},$$

then symplectically completing this basis in $\tilde{E}_\mu \oplus \tilde{E}_{\mu^{-1}}$ gives

$$A|_{\tilde{E}_{\mu^{-1}}} = \left(A_\mu^T\right)^{-1}.$$

As $A_\mu = \mu I + N_\mu$ with N_μ nilpotent, by expanding $\left(A_\mu^T\right)^{-1}$ as a power series, we see

$$A_{\mu^{-1}} := A|_{\tilde{E}_{\mu^{-1}}} = \mu^{-1} I + N_{\mu^{-1}}$$

with $N_{\mu^{-1}}$ nilpotent. We choose a branch of logarithm so that

$$\lambda(\mu) = \log(\mu)$$

satisfies

$$\lambda(\mu^{-1}) = -\lambda(\mu), \text{ and} \tag{6.1}$$

$$\lambda(\overline{\mu}) = \overline{\lambda(\mu)}, \tag{6.2}$$

and observe for N nilpotent,

$$\log(I+N) = N - \frac{N^2}{2} + \frac{N^3}{3} + \dots$$

is a finite series. Then we can define

$$\log(\mu I + N_\mu) = \lambda(\mu) + N_\lambda,$$

with N_λ nilpotent.

We apply this technique to each generalized eigenspace of A to obtain a complex matrix

$$\widetilde{B} := \log A.$$

We see $\widetilde{B}$ is block diagonal with diagonal elements of the form $\lambda I + N_\lambda$ with N_λ nilpotent. We know $|\mu| > 1$ real gives $\operatorname{Re}\lambda(\mu) > 0$. For λ satisfying $\operatorname{Re}\lambda > 0$, let E_λ denote the generalized complex eigenspace of λ with respect to $\widetilde{B}$, and let $\tilde{E}_{\tilde{\mu}}$ denote the generalized complex eigenspace of $\tilde{\mu}$ with respect to $dS(0,0)$. There are 4 cases to consider.

Case 1: $\tilde{\mu} > 1$ is real, and an eigenvalue of $dS(0,0)$. Then $E_\lambda \oplus E_{-\lambda}$ is a real symplectic space which is equal to $\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}}$. If we put $\widetilde{B}$ into Jordan form over E_λ,

$$\widetilde{B}_\lambda := \widetilde{B}\Big|_{E_\lambda} = \begin{pmatrix} \lambda & 1 & 0 & \dots\dots \\ 0 & \lambda & 1 & 0 \;\; \dots \\ \vdots & \ddots & \ddots & \dots\dots \\ 0 & \dots & 0 & \lambda \;\;\; 1 \\ 0 & \dots\dots\dots\dots & & \lambda \end{pmatrix},$$

completing the basis symplectically over $E_\lambda \oplus E_{-\lambda}$ gives

$$\left.\widetilde{B}\right|_{E_{-\lambda}} = -\left(\widetilde{B}_\lambda\right)^T.$$

As $\mu = \tilde{\mu}$ was an eigenvalue of A,

$$\exp(-JF)|_{\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}}} = \mathrm{id}\,.$$

Case 2: $\tilde{\mu}$ is complex, $|\tilde{\mu}| > 1$, and $\tilde{\mu}$ is an eigenvalue of $dS(0,0)$. Then $E_\lambda \oplus E_{-\lambda} \oplus E_{\bar{\lambda}} \oplus E_{-\bar{\lambda}}$ is the complexification of a real symplectic vector space which is equal to $\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}} \oplus \tilde{E}_{\overline{\tilde{\mu}}} \oplus \tilde{E}_{\overline{\tilde{\mu}}^{-1}}$. Changing variables as in [Chr1] §6, we see

$$\left.\widetilde{B}\right|_{E_\lambda \oplus E_{-\lambda} \oplus E_{\bar{\lambda}} \oplus E_{-\bar{\lambda}}} = \begin{pmatrix} \widetilde{B}_\lambda & 0 \\ 0 & -\left(\widetilde{B}_\lambda\right)^T \end{pmatrix},$$

where

$$\widetilde{B}_\lambda = \begin{pmatrix} \Lambda & I & 0 & \ldots\ldots & \\ 0 & \Lambda & I & 0 & \ldots \\ \vdots & \ddots & \ddots & \ldots\ldots & \\ 0 & \ldots & 0 & \Lambda & I \\ 0 & \ldots\ldots\ldots\ldots & & & \Lambda \end{pmatrix},$$

with I the 2×2 identity matrix and

$$\Lambda = \begin{pmatrix} \operatorname{Re}\lambda & -\operatorname{Im}\lambda \\ \operatorname{Im}\lambda & \operatorname{Re}\lambda \end{pmatrix}.$$

Further,

$$\exp(-JF)|_{\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}} \oplus \tilde{E}_{\overline{\tilde{\mu}}} \oplus \tilde{E}_{\overline{\tilde{\mu}}^{-1}}} = \mathrm{id}\,.$$

Case 3: $\mu > 1$ is real, and $\tilde{\mu} = -\mu$ is an eigenvalue of $dS(0,0)$. Then $E_\lambda \oplus E_{-\lambda}$ is a real symplectic vector space, equal to $\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}}$ and $\widetilde{B}$ is handled as in Case 1, with the important difference:

$$\exp(-JF)|_{\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}}} = -\,\mathrm{id}\,.$$

Case 4: $|\tilde{\mu}| = 1$, $\operatorname{Im}\tilde{\mu} > 0$ is an eigenvalue of $dS(0,0)$. Then $E_\lambda \oplus E_{-\lambda}$ is a complex symplectic vector space which is the complexification of a real symplectic vector space which is equal to $\tilde{E}_{\tilde{\mu}} \oplus \tilde{E}_{\tilde{\mu}^{-1}}$. Since we have assumed in particular that $\tilde{\mu}$ occurs with multiplicity 1, so does λ. Write $\lambda = i\alpha$, $\alpha > 0$, in which case we observe

$$F|_{E_{i\alpha} \oplus E_{-i\alpha}} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

is diagonal since $\tilde{\mu}$ is distinct.

We have proved the following proposition, which we record in detail to fix our notation.

Proposition 6.1. *Let $S : W_1 \to W_2$, $W_1, W_2 \subset \mathbb{R}^{2n-2}$ be a local symplectic map, $S(0,0) = (0,0)$, and let n_{hc} be the number of Jordan blocks of complex eigenvalues μ of $dS(0,0)$ satisfying $|\mu| > 1$, $\operatorname{Re}\mu > 1$, and $\operatorname{Im}\mu > 0$; n_{hr+} be the number of Jordan blocks of real positive eigenvalues μ of $dS(0,0)$ satisfying $\mu > 1$; n_{hr-} be the number of Jordan blocks of negative real eigenvalues $-\mu$ of $dS(0,0)$ satisfying*

$-\mu < -1$; and n_e be the number of eigenvalues μ of modulus 1 satisfying $\operatorname{Im}\mu > 0$. For

$$j \in (1, \ldots, n_{hc}; 2n_{hc}+1, \ldots, 2n_{hc}+n_{hr+}; \tag{6.3}$$

$$2n_{hc}+n_{hr+}+1, \ldots, 2n_{hc}+n_{hr+}+n_{hr-}; \tag{6.4}$$

$$2n_{hc}+n_{hr+}+n_{hr-}+1, \ldots, 2n_{hc}+n_{hr+}+n_{hr-}+n_e) \tag{6.5}$$

let k_j denote the multiplicity of μ_j so that

$$2n-2 = 4\left(\sum_{j=1}^{n_{hc}} k_j\right) + 2\left(\sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr+}} k_j\right) + 2\left(\sum_{2n_{hc}+n_{hr+}+1}^{2n_{hc}+n_{hr+}+n_{hr-}} k_j\right) + 2\left(\sum_{2n_{hc}+n_{hr+}+n_{hr-}+1}^{2n_{hc}+n_{hr+}+n_{hr-}+n_e} k_j\right).$$

Choose $\lambda_j(\mu_j) = \log\mu_j$ satisfying (6.1-6.2) for j in the range (6.3) and (6.5), and for j in the range (6.4), choose $\lambda_j(\mu_j) = \log(-\mu_j)$ satisfying (6.1-6.2). Then there are real matrices B and F satisfying ${}^{\omega}B = -B$, $F^* = F$, and a symplectic choice of coordinates such that

$$dS(0,0) = \exp(-JF)\exp(B),$$

and B is of the form

$$B = diag\left(B_j; -B_j^T\right),$$

for j in the range (6.3-6.5). For $j \in (1, \ldots n_{hc})$, B_j is the $2k_j \times 2k_j$ matrix

$$B_j = \begin{pmatrix} \Lambda_j & I & 0 & \ldots\ldots\ldots & \\ 0 & \Lambda_j & I & 0 & \ldots \\ \vdots & \ddots & \ddots & \ldots\ldots\ldots & \\ 0 & \ldots & 0 & \Lambda_j & I \\ 0 & \ldots\ldots\ldots & & & \Lambda_j \end{pmatrix}, \tag{6.6}$$

with I the 2×2 identity matrix and

$$\Lambda_j = \begin{pmatrix} \operatorname{Re}\lambda_j & -\operatorname{Im}\lambda_j \\ \operatorname{Im}\lambda_j & \operatorname{Re}\lambda_j \end{pmatrix}.$$

For $j \in (2n_{hc}+1, \ldots 2n_{hc}+n_{hr+}+n_{hr-})$, B_j is the $k_j \times k_j$ matrix

$$B_j = \begin{pmatrix} \lambda_j & 1 & 0 & \ldots\ldots\ldots & \\ 0 & \lambda_j & 1 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ldots\ldots\ldots & \\ 0 & \ldots & 0 & \lambda_j & 1 \\ 0 & \ldots\ldots\ldots & & & \lambda_j \end{pmatrix}, \tag{6.7}$$

and for $j \in (2n_{hc}+n_{hr+}+n_{hr-}+1, \ldots, 2n_{hc}+n_{hr+}+n_{hr-}+n_e)$, B_j is the 1×1 matrix 0. Here

$$F = diag\left(F_j; F_j\right),$$

for j in the range (6.3-6.5), where for $j \in (1, \ldots, 2n_{hc})$, F is the $2k_j \times 2k_j$ zero matrix, for $j \in (2n_{hc}+1, \ldots, 2n_{hc}+n_{hr+})$, F_j is the $k_j \times k_j$ zero matrix, for $j \in (2n_{hc}+n_{hr+}+1, \ldots 2n_{hc}+n_{hr+}+n_{hr-})$,

$$F_j = \pi I,$$

where I is the $k_j \times k_j$ identity matrix, and for $j \in (2n_{hc}+n_{hr+}+n_{hr-}+1, \ldots, 2n_{hc}+n_{hr+}+n_{hr-}+n_e)$, $F_j = \operatorname{Im} \lambda_j$.

Following the proof of [EvZw, Theorem 10.1], we set

$$K_t^1 = \exp(-tJF) \text{ and } K_t = \exp(tB),$$

which we observe is the same as

$$K_t^1 = \exp(tH_{q^1}) \text{ and } K_t = \exp(tH_q)$$

for

$$q(x,\xi) =$$

$$= \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j} \left(\operatorname{Re} \lambda_j \, (x_{2l-1}\xi_{2l-1} + x_{2l}\xi_{2l}) - \operatorname{Im} \lambda_j \, (x_{2l-1}\xi_{2l} - x_{2l}\xi_{2l-1}) \right) \tag{6.8}$$

$$+ \sum_{j=1}^{n_{hc}} \sum_{l=1}^{k_j-1} (x_{2l+1}\xi_{2l-1} + x_{2l+2}\xi_{2l}) \tag{6.9}$$

$$+ \sum_{j=2n_{hc}+1}^{2n_{hc}+n_{hr+}+n_{hr-}} \left(\sum_{l=1}^{k_j} \lambda_j x_l \xi_l + \sum_{l=1}^{k_j-1} x_{l+1}\xi_l \right), \tag{6.10}$$

and

$$q^1(x,\xi) =$$

$$\sum_{j=2n_{hc}+n_{hr+}+1}^{2n_{hc}+n_{hr+}+n_{hr-}} \frac{\pi}{2}(x_j^2 + \xi_j^2) + \sum_{j=2n_{hc}+n_{hr+}+n_{hr-}+1}^{2n_{hc}+n_{hr+}+n_{hr-}+n_e} \frac{\operatorname{Im} \lambda_j}{2}(x_j^2 + \xi_j^2). \tag{6.11}$$

6.2. **Geometry of the Poincaré Section.** The previous section motivates the next proposition. First we need the following lemma, which follows from the more general [Chr1, Lemma 4.2]. Recall under the assumption that S is hyperbolic, the stable and unstable manifolds $\Lambda_\mp \subset N$ for S are $n-1$-dimensional locally embedded transversal Lagrangian submanifolds (see [HaKa, Theorem 6.2.3]).

Lemma 6.2. *Let $S : W_1 \to W_2$, $W_1, W_2 \subset \mathbb{R}^{2n-2}$, $S(0,0) = (0,0)$, be a local hyperbolic symplectic map with unstable/stable manifolds $\Lambda_\pm$. Then there exists a local symplectic coordinate system (x,ξ) near γ such that $\Lambda_+ = \{\xi = 0\}$ and $\Lambda_- = \{x = 0\}$.*

For the following proposition, we assume there are no negative real eigenvalues and no eigenvalues of modulus 1 to the linearized Poincaré map. Later we will modify the general Poincaré map to be of this form. This follows from the proof of [Chr1, Proposition 4.3].

Proposition 6.3. *Let $S : W_1 \to W_2$, $W_1, W_2 \subset \mathbb{R}^{2n-2}$, be a local hyperbolic symplectic map, $S(0,0) = (0,0)$, and assume $dS(0,0)$ has no negative real eigenvalues.*

There is a smooth family of local symplectomorphisms κ_t, a smooth, real-valued matrix function $B_t(x,\xi)$, and a symplectic choice of coordinates in which

$$\begin{aligned} &\text{(i) } \kappa_0 = \mathrm{id}\,, \;\; \kappa_1(x,\xi) = S(x,\xi); \\ &\text{(ii) } \frac{d}{dt}\kappa_t = (\kappa_t)_* \, H_{q_t}, \end{aligned} \tag{6.12}$$

where

$$q_t(x,\xi) = \langle B_t(x,\xi)x, \xi\rangle \,. \tag{6.13}$$

Here

$$\langle B_t(0,0)x,\xi\rangle = q(x,\xi), \tag{6.14}$$

for $q(x,\xi)$ of the form (6.8-6.10).

7. The Proof of Theorem 1

7.1. **Motivation.** We recall from Theorem 4 that if $u \in L^2(X)$ has wavefront set sufficiently close to γ and $B \in \Psi^{0,0}(V)$ is a microlocal cutoff near $(0,0)$, we have for $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$,

$$\|(P-z)u\|_{L^2(X)} \geq C^{-1} h \left\|B(I - M(z))R_+ u\right\|_{L^2(V)} .$$

Hence we want to show $M(z)$ has spectrum away from 1. This is the content of the following Theorem, which we state in its general form for reference.

Theorem 6. *Let $\widetilde{V} \subset \mathbb{R}^{2m}$ be an open neighbourhood of $(0,0)$, and assume κ_z : $\mathrm{neigh}\,(\widetilde{V}) \to \kappa_z(\mathrm{neigh}\,(\widetilde{V}))$, $\kappa_z(0,0) = (0,0)$, $z \in (-\delta,\delta)$, $\delta > 0$ is a smooth family of symplectomorphisms such that $d\kappa_z(0,0)$ is semi-hyperbolic and the nonresonance condition* (1.2) *holds for $d\kappa_z(0,0)$. Let $M(z)$ be the microlocally unitary h-FIO which quantizes κ_z as in* [EvZw, Theorem 10.3]*. Then for $z \in (-\delta',\delta')$, $\delta' > 0$ sufficiently small and $s \in \mathbb{R}$ sufficiently close to* 0*, there exist self-adjoint, semi-classically tempered operators* $\exp(\pm s K^w)$ *so that for $v \in L^2(\mathbb{R}^m)$ with h-wavefront set sufficiently close to* $(0,0)$,

$$\left\| e^{-sK^w} M(z) e^{sK^w} v \right\|_{L^2} \leq \frac{1}{R}\|v\|_{L^2}. \tag{7.1}$$

From §3, we know $M(z)$ is an h-FIO associated to the graph of $S(z)$, where $S(z)$ is the Poincaré map for γ_z, the periodic orbit in the energy level z. Suppose for the moment that $S(z)$ satisfies the hypotheses of Proposition 6.3, and let $q_{z,t}$ be q_t as in the conclusion of the Proposition, where now $q_{z,t}$ varies over energy levels z near 0. Setting $Q_{z,t} = \mathrm{Op}_h^w(q_{z,t})$, by modifying the proofs of [EvZw, Theorems 10.3, 10.7], there exists $M_{z,0} \in \Psi_h^{0,0}$ microlocally unitary so that $M(z) = M^z(1)$ for $M^z(t)$ a family of operators satisfying the evolution equations

$$\begin{aligned} hD_t M^z + M^z Q_{z,t} &= 0, \;\; 0 \leq t \leq 1, \\ M^z(0) &= M_{z,0}. \end{aligned}$$

In order to prove Theorem 6, we observe if $W(z) : L^2(V) \to L^2(V)$ is the microlocal inverse for $M(z)$, we have also $W_{z,0} \in \Psi_h^{0,0}$ microlocally unitary so that $W(z) = W^z(1)$ for $W^z(t)$ satisfying the following evolution equation:

$$hD_t W^z - Q_{z,t} W^z = 0, \;\; 0 \leq t \leq 1, \tag{7.2}$$

$$W^z(0) = W_{z,0}. \tag{7.3}$$

The rest of this section is devoted to proving there exist semiclassically tempered operators $\exp(\pm sK^w)$ as in the statement of the Theorem so that

$$\left\| e^{-sK^w} W(z) e^{sK^w} v \right\|_{L^2(V)} \geq R \|v\|_{L^2(V)} \tag{7.4}$$

for some $R > 1$. Then

$$\begin{aligned} \|v\|_{L^2(V)} &= \left\| e^{-sK^w} W(z) e^{sK^w} e^{-sK^w} M(z) e^{sK^w} v \right\|_{L^2(V)} \\ &\geq R \left\| e^{-sK^w} M(z) e^{sK^w} v \right\|_{L^2(V)}, \end{aligned}$$

which gives the Theorem once we prove (7.4).

In order to get Theorem 1 from Theorem 6, we observe by (7.1) we have also

$$\left\| \left(I - e^{-sK^w} M(z) e^{sK^w} \right) v \right\| \geq C^{-1} \|v\|.$$

Thus

$$\begin{aligned} \operatorname{Re} \left\langle e^{-sK^w} (I - M) e^{sK^w} v, v \right\rangle &= \|v\|^2 - \operatorname{Re} \left\langle e^{-sK^w} M(z) e^{sK^w} v, v \right\rangle \\ &\geq C^{-1} \|v\|^2_{L^2(V)}. \end{aligned}$$

Since $\| \exp(\pm sK^w) \|_{L^2 \to L^2} = \mathcal{O}(h^{-N})$ for some N, we have

$$\operatorname{Re} \langle (I - M(z)) v, v \rangle \geq C h^N \|v\|^2.$$

Now let $u \in L^2(X)$ have wavefront set close to γ. Set $v = R_+ u$ so that $\mathrm{WF}_h v$ is close to $(0,0)$, and observe with B as in Theorem 4 and $b = \sigma_h(B)$, $1 - b$ has support away from $(0,0) \in T^*\mathbb{R}^{n-1}$. Then

$$\operatorname{Op}_h^w(1-b) M(z) v = M(z) \operatorname{Op}_h^w(S(z)^*(1-b)) v = \mathcal{O}(h^\infty),$$

so that if $\mathrm{WF}_h u$ is sufficiently small,

$$B(I - M(z)) R_+ u = (I - M(z)) R_+ u$$

microlocally, and (4.12) gives the theorem.

□

Our biggest tool so far is the normal form deformation in Proposition 6.3, however we cannot immediately apply it to $S(z)$ satisfying the assumptions of the introduction. To get by this we will transform $S(z)$ into a hyperbolic map satisfying the assumptions of Proposition 6.3 and then later deal with the errors which come up when transforming back.

The proof of Theorem 6 will proceed in 4 basic steps. First, we deform the effective Hamiltonian into a sum of two Hamiltonians with disjoint support in t, one hyperbolic and one elliptic. The summed Hamiltonian will be called $q_{z,t}$. We then modify the evolution equation defining W^z to an equation involving a conjugated version of W^z, $\widetilde{W}(t)$. This evolution equation will be given in terms of a conjugated quantization of $q_{z,t}$, $\widetilde{Q}_{z,t}$, that we will then need to estimate from below. This step is a variation on the classical idea of a "positive commutator". That is, $\operatorname{Op}_h^w(q_{z,t})$ is self-adjoint, but if we conjugate it with an operator of the form e^{G^w}, we get $\operatorname{Op}_h^w(q_{z,t})$ plus a lower order *skew*-adjoint commutator. The principal symbol of the commutator $[G, \operatorname{Op}(q_{z,t})]$ is $hiH_{q_{z,t}}G$. The linear part of $H_{q_{z,t}}$ is block diagonal in the hyperbolic and elliptic variables, but the nonlinear part potentially forces interaction between the hyperbolic and elliptic variables. Hence we will be forced to introduce a complex weight G to gain some orthogonality between the hyperbolic

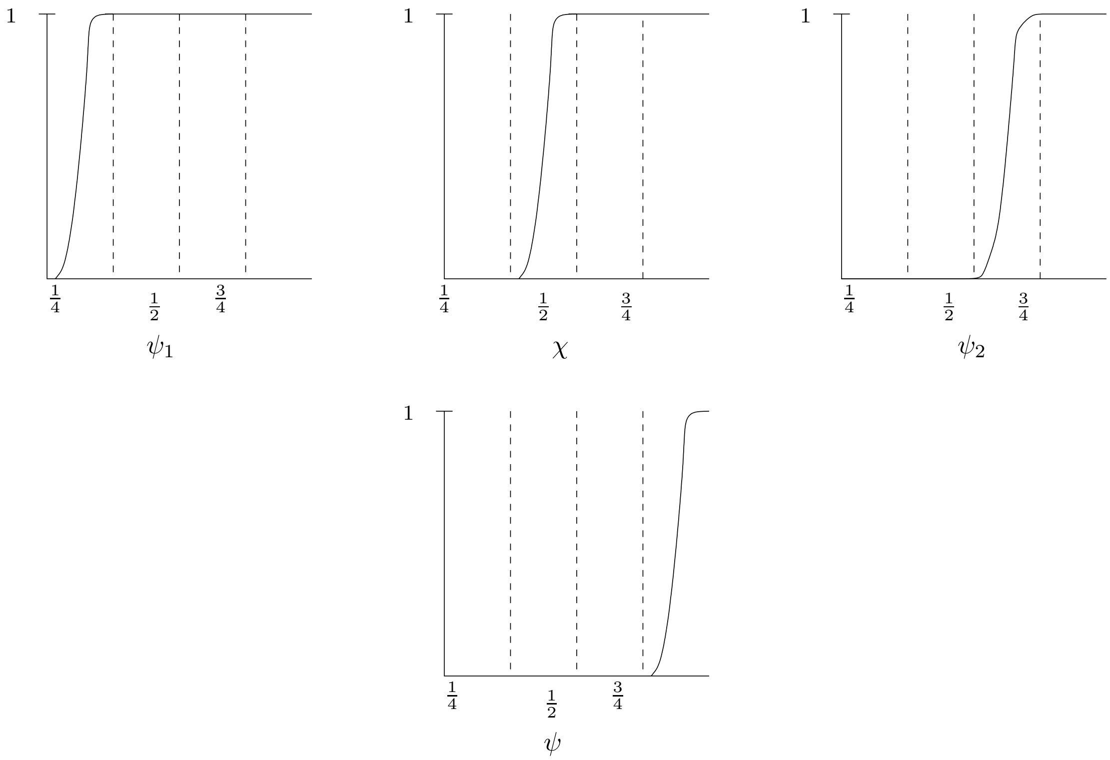

FIGURE 7. The cutoff functions ψ_1, ψ_2, ψ, and χ.

and elliptic variables. This is accomplished in Step 3. Finally we will estimate M, whose inverse is related to $\widetilde{W}$ by conjugation.

7.2. **Step 1: Deform $q_{z,t}$.** We construct a rescaled deformation of identity into $S(z)$ in which the elliptic part of the effective Hamiltonian has disjoint support in t from the support of the non-elliptic part.

We will be using four cutoff functions,

$$\psi_1(t), \psi_2(t), \psi(t), \text{ and } \chi(t) : [0,1] \to [0,1]$$

satisfying the following properties (see Figure 7):

(i) $\psi_1(0) = \psi_2(0) = \psi(0) = \chi(0) = 0, \;\; \psi_1(1) = \psi_2(1) = \psi(1) = \chi(1) = 1;$
(ii) $\psi_1', \; \psi'$, and χ' are all non-negative,
(iii) $\operatorname{supp} \psi_1' \subset [0, 1/4], \; \operatorname{supp} \chi' \subset [1/4, 1/2],$
$\operatorname{supp} \psi' \subset [1/2, 3/4]$, and $\operatorname{supp} \psi' \subset [3/4, 1]$.

Motivated by Proposition 6.3, we construct a family of symplectomorphisms, $\kappa_{z,t}$, satisfying $\kappa_{z,0} = \mathrm{id}$ and $\kappa_{z,1} = S(z)$, but the elliptic part has disjoint support in t from the hyperbolic part. That is, let F be given as in Proposition 6.1, and let

$$E(z) = \exp(-JF(z)), \;\; K_t^1 = \exp(-tJF(z)),$$

so that $K_0^1 = \mathrm{id}$, $K_1^1 = E(z)$, and

$$\frac{d}{dt} K_t^1 = (K_t^1)_* H_{q^1},$$

where q^1 is given by (6.11). Here the coefficients $\frac{\operatorname{Im} \lambda_j}{2}$ implicitly depend on z, but the dimensions of the eigenspaces are constant for z in a neighbourhood of 0. Let $\widetilde{K}_t^1$ be defined by

$$\widetilde{K}_t^1 = K^1_{\psi_1(t)},$$

so that $\widetilde{K}_0^1 = \mathrm{id}$, $\widetilde{K}_1^1 = E(z)$, and the chain rule then gives

$$\begin{aligned}
\frac{d}{dt} \widetilde{K}_t^1 &= \psi_1'(t) \frac{d}{d\tau} K_\tau^1 |_{\tau = \psi_1(t)} \\
&= \psi_1'(t) (K_\tau^1)_* H_{q^1} |_{\tau = \psi_1(t)} \\
&= (\widetilde{K}_t^1)_* H_{\psi_1'(t) q^1}.
\end{aligned}$$

We introduce an "artificial hyperbolic" transformation which will temporarily replace the elliptic part by setting

$$q_{ah} = \sum_{j = 2h_{hc} + n_{hr+} + n_{hr-} + 1}^{2h_{hc} + n_{hr+} + n_{hr-} + n_e} 2 x_j \xi_j,$$

defining $K_{ah} = \exp(H_{q_{ah}})$, and

$$\widetilde{S}(z) = K_{ah}^{-1} \circ E(z)^{-1} \circ S(z),$$

so that $\widetilde{S}(z)$ satisfies the assumptions of Proposition 6.3 near $z = 0$. From Proposition 6.3, there is a family $\kappa^1_{z,t}$ satisfying $\kappa^1_{z,0} = \mathrm{id}$, $\kappa^1_{z,1} = \widetilde{S}(z)$, and

$$\frac{d}{dt} \kappa^1_{z,t} = (\kappa^1_{z,t})_* H_{\tilde{q}_{z,t}},$$

where now

$$\tilde{q}_{z,t} = \langle B_{z,t}(x, \xi) x, \xi \rangle$$

for $B_{z,t}$ satisfying (6.14). Let

$$\tilde{\kappa}_{z,t} = \kappa^1_{z, \psi(t)},$$

so that $\tilde{\kappa}_{z,0} = \mathrm{id}$, $\tilde{\kappa}_{z,1} = \widetilde{S}(z)$, and

$$\begin{aligned}
\frac{d}{dt} \tilde{\kappa}_{z,t} &= \psi'(t) \frac{d}{d\tau} \kappa^1_{z,\tau} |_{\tau = \psi(t)} \\
&= \psi'(t) (\kappa^1_{z,\tau})_* H_{\tilde{q}_{z,\tau}} |_{\tau = \psi(t)} \\
&= (\tilde{\kappa}_{z,t})_* H_{\psi'(t) \tilde{q}_{z,\psi(t)}}.
\end{aligned}$$

Let $K_t^2 = \exp(t H_{q_{ah}})$ and $\widetilde{K}_t^2 = K^2_{\psi_2(t)}$, so that $\widetilde{K}_0^2 = \mathrm{id}$, $\widetilde{K}_1^2 = K_{ah}$, and

$$\frac{d}{dt} \widetilde{K}_t^2 = (\widetilde{K}_t^2)_* H_{\psi_2'(t) q_{ah}}.$$

Finally, let

$$\kappa_{z,t} = \widetilde{K}_t^1 \circ \widetilde{K}_t^2 \circ \tilde{\kappa}_{z,t}.$$

Unraveling the definitions, we have $\kappa_{z,t}$ satisfying

$$\text{(i) } \kappa_{z,0} = \mathrm{id}\,,\ \ \kappa_{z,1} = S(z);$$

$$\text{(ii) } \kappa_{z,t} = \begin{cases} \widetilde{K}_t^1,\ \ 0 \leq t \leq 1/4; \\ E(z),\ \ 1/4 \leq t \leq 1/2; \\ E(z) \circ \widetilde{K}_t^2,\ \ 1/2 \leq t \leq 3/4; \\ E(z) \circ K_{ah} \circ \tilde{\kappa}_{z,t}. \end{cases}$$

If we compose a smooth function a with $\kappa_{z,t}$, we have

$$\begin{aligned} \frac{d}{dt}\kappa_{z,t}^* a \quad &= \quad \begin{cases} \frac{d}{dt}a(\widetilde{K}_t^1),\ \ 0 \leq t \leq 1/4; \\ \frac{d}{dt}a(E(z)),\ \ 1/4 \leq t \leq 1/2; \\ \frac{d}{dt}a(E(z)) \circ \widetilde{K}_t^2,\ \ 1/2 \leq t \leq 3/4; \\ \frac{d}{dt}a(E(z)K_{ah}) \circ \tilde{\kappa}_{z,t},\ \ 3/4 \leq t \leq 1; \end{cases} \\ &= \quad \begin{cases} (H_{\psi_1'(t)q^1} a) \circ \widetilde{K}_t^1,\ \ 0 \leq t \leq 1/4; \\ 0,\ \ 1/4 \leq 1/2; \\ \big(H_{(E(z)^{-1})^*\psi_2'(t)q_{ah}} a\big) \circ E(z) \circ \widetilde{K}_t^2,\ \ 1/2 \leq t \leq 3/4; \\ \Big(H_{(K_{ah}^{-1})^*(E(z)^{-1})^*\psi'(t)\tilde{q}_{z,\psi(t)}} a\Big) \circ E(z) \circ K_{ah} \circ \tilde{\kappa}_{z,t},\ \ 3/4 \leq t \leq 1. \end{cases} \end{aligned}$$

Summing up and using the support properties of ψ, ψ_1, and ψ_2, we have

$$\frac{d}{dt}\kappa_{z,t} = (\kappa_{z,t})_* H_{\tilde{q}_{z,t}^2},$$

where

$$\tilde{q}_{z,t}^2 = (E(z)^{-1}K_{ah}^{-1})^*\psi'(t)\tilde{q}_{z,\psi(t)} + (E(z)^{-1})^*\psi_1'(t)q^1 + \psi_2'(t)q_{ah}. \tag{7.5}$$

We record for later use that since $\psi = \psi' = 0$ and $\psi_2 = \psi_2' = 0$ on the support of χ', we have for $t \in \operatorname{supp}\chi'$,

$$\kappa_{z,t} = E(z) = \begin{pmatrix} \mathrm{id}_{\,h+} & 0 & 0 \\ 0 & -\mathrm{id}_{\,h-} & 0 \\ 0 & 0 & \tilde{E(z)} \end{pmatrix}, \tag{7.6}$$

where $\mathrm{id}_{\,h+}$ is identity in x_j and ξ_j for

$$1 \leq j \leq 2n_{hc} + n_{hr+},$$

$\mathrm{id}_{\,h-}$ is the identity in x_j and ξ_j for

$$2n_{hc} + n_{hr+} + 1 \leq j \leq 2n_{hc} + n_{hr+} + n_{hr-},$$

and $\tilde{E(z)}$ is a z-dependent family of elliptic symplectic transformation in the variables x_j and ξ_j for

$$2n_{hc} + n_{hr+} + n_{hr-} + 1 \leq j \leq 2n_{hc} + n_{hr+} + n_{hr-} + n_e.$$

7.3. **Step 2: Conjugation of Evolution Equations.** For Step 2, we introduce the following notation. By $(X_{\mathrm{hyp}}, \Xi_{\mathrm{hyp}})$ and $(X_{\mathrm{ell}}, \Xi_{\mathrm{ell}})$ we mean the symplectic variables in the subspace associated to the hyperbolic and elliptic parts of $dS(0,0)$ respectively. In our notation,

$$X_{\mathrm{hyp}} = (X_1, \ldots, X_{n-n_e-1}),\ \ \Xi_{\mathrm{hyp}} = (\Xi_1, \ldots, \Xi_{n-n_e-1}),$$

and

$$X_{\mathrm{ell}} = (X_{n-n_e}, \ldots, X_{n-1}),\ \ \Xi_{\mathrm{ell}} = (\Xi_{n-n_e}, \ldots, \Xi_{n-1}).$$

The assumption that γ is semi-hyperbolic amounts to saying $n-1-n_e \geq 1$, or that X_{hyp} are non-trivial variables. For a vector $Y \in \mathbb{R}^{n-1}$, we define also

$$\begin{aligned} |Y|^2_{\mathrm{hyp}} &= \sum_{j=1}^{n-n_e-1} Y_j^2 \text{ and} \\ |Y|^2_{\mathrm{ell}} &= \sum_{j=n-n_e}^{n-1} Y_j^2, \end{aligned}$$

where as usual $n_e = n-1-2n_{hc}-n_{hr+}-n_{hr-}$. If

$$\langle B\cdot, \cdot\rangle : \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{C}$$

is a bilinear form, we will also use the notation

$$\langle BY, (Z_{\mathrm{hyp}}, iZ_{\mathrm{ell}})\rangle = \sum_{j=1}^{n-1}\sum_{k=1}^{n-n_e-1} B^{jk}Y_jZ_k + i\sum_{j=1}^{n-1}\sum_{k=n-n_e}^{n-1} B^{jk}Y_jZ_k.$$

Let $W(z) = M(z)^{-1}$ as above and let $Q_{z,t} = \mathrm{Op}\,_h^w(\tilde{q}^2_{z,t})$ for $\tilde{q}^2_{z,t}$ in the form (7.5). Again by modifying the proofs of [EvZw, Theorems 10.3, 10.7], there is $W^z(t)$ and $W_{z,0}$ unitary satisfying (7.2-7.3) with this choice of $Q_{z,t}$ so that $W^z(1) = W(z)$. As in §5, but with W^z instead of M^z, if we conjugate $W^z(t)$ satisfying (7.2-7.3) in a way which is independent of t, we get a new equation with a conjugated $Q_{z,t}$. That is, with $T_{h,\tilde{h}}$ defined in (5.1), let

$$W^{z,1}(t) = T_{h,\tilde{h}} W^z(t) T^{-1}_{h,\tilde{h}}$$

and observe $W^{z,1}(t)$ satisfies

$$\begin{aligned} hD_tW^{z,1} - Q^1_{z,t}W^{z,1} &= 0, \;\; 0 \leq t \leq 1 \\ W^{z,1}(0) &= T_{h,\tilde{h}} W_{z,0} T^{-1}_{h,\tilde{h}} \end{aligned}$$

for $Q^1_{z,t} = T_{h,\tilde{h}} Q_{z,t} T^{-1}_{h,\tilde{h}}$.

We define the escape function G in the new coordinates by

$$\begin{aligned} G(X,\Xi) &= \frac{1}{2}\log\left(\frac{1+|X|^2_{\mathrm{hyp}}}{1+|\Xi|^2_{\mathrm{hyp}}}\right) + i\frac{1}{2}(|X_{\mathrm{ell}}|^2 - |\Xi_{\mathrm{ell}}|^2) \\ &=: G_1 + iG_2. \end{aligned} \tag{7.7}$$

Here we have added an imaginary term to the definition of G. Observe

$$\exp(i\mathrm{Op}\,_{\tilde{h}}^w(G_2))$$

is unitary. As mentioned in the introduction to this section, this is used to control the nonlinear interactions between the hyperbolic and elliptic variables in a Poisson bracket later in the proof.

The real part of G, G_1, satisfies

$$\left|\partial_X^\alpha \partial_\Xi^\beta G_1(X,\Xi)\right| \leq C_{\alpha\beta}\langle X\rangle^{-|\alpha|}\langle\Xi\rangle^{-|\beta|}, \quad \text{for } (\alpha,\beta)\neq(0,0),$$

and since $\langle X\rangle^2\langle\Xi\rangle^{-2}$ is an order function, $\mathrm{Re}\,G$ satisfies the assumptions of [Chr1, Lemma 2.1]. Thus we can construct the operators $e^{\pm s\chi(t)G^w}$, where G^w is the $\tilde{h}$-Weyl quantization of G, and doing so we may define

$$\widetilde{W}(t) = e^{-s\chi G^w} W^{z,1}(t) e^{s\chi G^w}. \tag{7.8}$$

Similar to §5, $\widetilde{W}$ satisfies the evolution equation

$$hD_t\widetilde{W} - \widetilde{Q}_{z,t}\widetilde{W} = \frac{h}{i}s\chi'(t)e^{-s\chi G^w}\left[W^{z,1}, G^w\right]e^{s\chi G^w}, \quad 0 \leq t \leq 1 \tag{7.9}$$

$$\widetilde{W}(0) = e^{-s\chi(0)G^w}T_{h,\tilde{h}}W_{z,0}T_{h,\tilde{h}}^{-1}e^{s\chi(0)G^w}, \tag{7.10}$$

where

$$\widetilde{Q}_{z,t} = e^{-s\chi G^w}Q^1_{z,t}e^{s\chi G^w}.$$

The definition of $W^{z,1}$ together with modifying the proof of [EvZw, Theorem 10.3] to the 2-parameter setting (using [Chr1, Lemma 2.5] to estimate the commutators) implies

$$\begin{aligned}\chi'(t)\left[W^{z,1}, G^w\right] &= \chi'(t)T_{h,\tilde{h}}\left[W^z, T_{h,\tilde{h}}^{-1}G^wT_{h,\tilde{h}}\right]T_{h,\tilde{h}}^{-1}\\ &= \chi'(t)T_{h,\tilde{h}}\mathrm{Op}_h^w\left(\kappa^*_{z,t}\widetilde{G} - \widetilde{G} + \mathcal{O}(h^{1/2}\tilde{h}^{3/2})\right)W^zT_{h,\tilde{h}}^{-1},\end{aligned}$$

where

$$\widetilde{G}(x,\xi) = G\left((\tilde{h}/h)^{\frac{1}{2}}(x,\xi)\right) \in \mathcal{S}^{-\infty,0,0}_{\frac{1}{2}} \text{ microlocally.}$$

From (7.6) and the definition of G,

$$\mathrm{Re}\,\kappa^*_{z,t}\widetilde{G} = \mathrm{Re}\,\widetilde{G}$$

on $\mathrm{supp}\,\chi'$. Hence, using [Chr1, Lemma 2.5] and the modification of [EvZw, Theorem 10.3] to the 2-parameter setting, there is a symbol $e_t \in \mathcal{S}_0^{-\infty,-1/2,-3/2}$ such that

$$\begin{aligned}&\mathrm{Im}\,\frac{h}{i}s\chi'(t)e^{-s\chi G^w}\left[W^{z,1}, G^w\right]e^{s\chi G^w} =\\ &= \mathrm{Im}\,\frac{h}{i}s\chi'(t)\left(\mathrm{Op}_{\tilde{h}}(e_t) + \frac{\tilde{h}}{i}s\chi'(t)G^w\mathrm{Op}_{\tilde{h}}(\{e_t, G\}) + \mathcal{O}(h^{1/2}\tilde{h}^{7/2})\right)\\ &= \mathcal{O}(h^{3/2}\tilde{h}^{3/2}).\end{aligned}$$

7.4. **Step 3: Estimation of $\widetilde{Q}_{z,t}$.** We want to gain some knowledge of $\widetilde{Q}_{z,t}$. For that we use the techniques from the proof of Theorem 1 in [Chr1] together with the necessary modifications discussed in the introduction. We summarize the content of this Step in the following Lemma:

Lemma 7.1. *For $\widetilde{Q}_{z,t}$ as defined above, we have the estimate*

$$-\mathrm{Im}\,\langle\widetilde{Q}_{z,t}u, u\rangle \geq \psi'(t)\frac{h\tilde{h}}{C}\|u\|^2, \tag{7.11}$$

for any $u \in L^2(\mathbb{R}^{n-1})$.

The idea is that the conjugated $\widetilde{Q}_{z,t}$ is $Q^1_{z,t}$ to leading order, which is self-adjoint, and the second order term is roughly the quantization of

$$\frac{\tilde{h}}{i}H_qG$$

for a quadratic form q. But then [Chr1, Theorem 4] and says that for the quadratic forms in which we are interested we can make H_qG into a positive definite quadratic form, and there are linear symplectic coordinates in which H_qG is almost the harmonic oscillator $\sum_j x_j^2 + \xi_j^2$.

Let U be a neighbourhood of $(0,0)$, $U \subset T^*\mathbb{R}^{n-1}$, and assume

$$U \subset U_{\epsilon/2} := \left\{ (x,\xi) : |(x,\xi)| < \frac{\epsilon}{2} \right\}$$

for $\epsilon > 0$. We assume throughout that we are working microlocally in U_ϵ. With $\tilde{h}$ small (fixed later in the proof), we have done the following rescaling:

$$X := \left(\tilde{h}/h\right)^{\frac{1}{2}} x, \; \Xi := \left(\tilde{h}/h\right)^{\frac{1}{2}} \xi \tag{7.12}$$

We assume for the remainder of the proof that $|(X,\Xi)| \leq \left(\tilde{h}/h\right)^{\frac{1}{2}} \epsilon$. We used the unitary operator $T_{h,\tilde{h}}$ defined in (5.1) to introduce the second parameter into $Q_{z,t}$ to get

$$Q^1_{z,t} = T_{h,\tilde{h}} Q_{z,t} T^{-1}_{h,\tilde{h}}$$

as above. On the support of $\chi(t)$, after a linear symplectic change of variables, we write

$$Q^2_{z,t} = T_{h,\tilde{h}} \mathrm{Op}\,_h(\psi'(t)\tilde{q}_{z,\psi(t)} + (K_{ah}E(z))^* \psi_2'(t) q_{ah}) T^{-1}_{h,\tilde{h}},$$

where $\tilde{q}_{z,t} = \langle B_{z,t} x, \xi \rangle$ defined in Step 1. The principal symbol of $Q^2_{z,t}$ on $\mathrm{supp}\,\chi'$ is

$$\begin{aligned} & q^2_{z,t}(X,\Xi) \\ = \; & q^3_{z,t}(X,\Xi) + q_{ah}((h/\tilde{h})^{\frac{1}{2}}(X,\Xi)) \\ = \; & \psi'(t) \left\langle B_{z,\psi(t)} \left(\left(h/\tilde{h}\right)^{\frac{1}{2}} (X,\Xi) \right) \left(h/\tilde{h}\right)^{\frac{1}{2}} X, \left(h/\tilde{h}\right)^{\frac{1}{2}} \Xi \right\rangle \\ & + \psi_2'(t) \sum_{j=2h_{hc}+n_{hr+}+n_{hr-}+1}^{2h_{hc}+n_{hr+}+n_{hr-}+n_e} (h/\tilde{h}) 2 X_j \Xi_j, \end{aligned} \tag{7.13}$$

and $q^2_{z,t} \in \mathcal{S}^{-\infty,0,0}_{-\frac{1}{2}}$ microlocally. We have

$$\left| \partial^\alpha_{X,\Xi} q^2_{z,t} \right| \leq C_\alpha \left(h/\tilde{h}\right)^{|\alpha|/2} \tag{7.14}$$

for $(X,\Xi) \in U_{(\tilde{h}/h)^{\frac{1}{2}}\epsilon}$ by [Chr1, Lemma 2.4].

Now $\mathrm{Re}\,\{G, q_{ah}((h/\tilde{h})^{1/2}(X,\Xi))\} = 0$, so to find the real part of $H_{q^2_{z,t}} G$, we need only calculate $\mathrm{Re}\, H_{q^3_{z,t}} G$. For $|(X,\Xi)| \leq \left(\tilde{h}/h\right)^{\frac{1}{2}} \epsilon$ we have with G as above in (7.7)

$$H_{q^3_{z,t}} G(X,\Xi) =$$

$$= \left(h/\tilde{h}\right) \psi'(t) \left[\left\langle B_{z,\psi(t)} X, \frac{\partial}{\partial X} \right\rangle - \left\langle B_{z,\psi(t)} \frac{\partial}{\partial \Xi}, \Xi \right\rangle \right] G(X,\Xi) \tag{7.15}$$

$$+ \left(h/\tilde{h}\right)^{\frac{3}{2}} \psi'(t) \left[\sum_{j=1}^{n-1} \left\langle \frac{\partial}{\partial \Xi_j} B_{z,\psi(t)}(\cdot,\cdot) X, \Xi \right\rangle \frac{\partial}{\partial X_j} G(X,\Xi) \right] \tag{7.16}$$

$$- \left(h/\tilde{h}\right)^{\frac{3}{2}} \psi'(t) \left[\sum_{j=1}^{n-1} \left\langle \frac{\partial}{\partial X_j} B_{z,\psi(t)}(\cdot,\cdot) X, \Xi \right\rangle \frac{\partial}{\partial \Xi_j} G(X,\Xi) \right]. \tag{7.17}$$

Now owing to Lemma [Chr1, Lemma 2.5] and (7.14) we have microlocally to leading order in h:

$$\operatorname{Re}\operatorname{ad}_{G^w}^{k}\left(Q_{z,t}^2\right)=\mathcal{O}_{L^2\to L^2}\left(h\tilde{h}^{k-1}\right),$$

and in particular,

$$i\operatorname{Im}\left[Q_{z,t}^2,G^w\right]=-i\tilde{h}\operatorname{Re}\operatorname{Op}_{\tilde{h}}^{w}\left(H_{q_{z,t}^2}G\right)+\mathcal{O}(h^{3/2}\tilde{h}^{3/2}). \tag{7.18}$$

Estimating the real part of the errors (7.16-7.17), we get

$$\begin{aligned}&\operatorname{Re}\left(h/\tilde{h}\right)^{\frac{3}{2}}\left[\sum_{j=1}^{n-1}\left\langle\frac{\partial}{\partial\Xi_j}B_{z,\psi(t)}(\cdot,\cdot)X,\Xi\right\rangle\frac{\partial}{\partial X_j}G(X,\Xi)\right]\\&\qquad=\left(h/\tilde{h}\right)^{\frac{3}{2}}\frac{1}{1+|X_{\text{hyp}}|^2}\mathcal{O}(|\Xi||X||X_{\text{hyp}}|),\end{aligned}\tag{7.19}$$

and analogously for (7.17). At $(0,0)$, $B_{z,\psi(t)}$ is positive definite and block diagonal of the form (6.14), so we compute:

$$\begin{aligned}\left|\left\langle B_{\psi(t)}(0,0)X,\left(\frac{X_{\text{hyp}}}{1+|X_{\text{hyp}}|^2},iX_{\text{ell}}\right)\right\rangle\right| &\geq C^{-1}\left(\frac{|X_{\text{hyp}}|^2}{1+|X_{\text{hyp}}|^2}+|X_{\text{ell}}|^2\right)\\&= C^{-1}\frac{|X|^2+|X_{\text{hyp}}|^2|X_{\text{ell}}|^2}{1+|X_{\text{hyp}}|^2}.\end{aligned}$$

Hence

$$\begin{aligned}&\operatorname{Re}\frac{1}{1+|X_{\text{hyp}}|^2}\mathcal{O}(|\Xi||X||X_{\text{hyp}}|)\frac{\left\langle B_{\psi(t)}(0,0)X,\left(\frac{X_{\text{hyp}}}{1+|X_{\text{hyp}}|^2},iX_{\text{ell}}\right)\right\rangle}{\left|\left\langle B_{\psi(t)}(0,0)X,\left(\frac{X_{\text{hyp}}}{1+|X_{\text{hyp}}|^2},iX_{\text{ell}}\right)\right\rangle\right|}\\&\quad=\operatorname{Re}\left\langle B_{\psi(t)}(0,0)X,\left(\frac{X_{\text{hyp}}}{1+|X_{\text{hyp}}|^2},iX_{\text{ell}}\right)\right\rangle\mathcal{O}(|\Xi|),\end{aligned}\tag{7.20}$$

and analogously for (7.17). Now we expand $B_{z,\psi(t)}$ in a Taylor approximation about $(0,0)$ to get

$$\begin{aligned}&\operatorname{Re}H_{q_{z,t}^3}G=\\&=\operatorname{Re}\left(h/\tilde{h}\right)\psi'(t)\Bigg[\left\langle B_{z,\psi(t)}(0,0)X,\left(\frac{X_{\text{hyp}}}{1+|X|_{\text{hyp}}^2},iX_{\text{ell}}\right)\right\rangle\\&\qquad+\operatorname{Re}\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}\left(\frac{|X||X_{\text{hyp}}|}{1+|X_{\text{hyp}}|^2}|(X,\Xi)|\right)\Bigg]\\&\quad+\operatorname{Re}\left(h/\tilde{h}\right)\psi'(t)\Bigg[\left\langle B_{z,\psi(t)}(0,0)\Xi,\left(\frac{\Xi_{\text{hyp}}}{1+|\Xi|_{\text{hyp}}^2},i\Xi_{\text{ell}}\right)\right\rangle\\&\qquad+\operatorname{Re}\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}\left(\frac{|\Xi||\Xi_{\text{hyp}}|}{1+|\Xi_{\text{hyp}}|^2}|(X,\Xi)|\right)\Bigg],\end{aligned}$$

which, from (7.20), is

$$\begin{aligned}
&\operatorname{Re} H_{q^3_{z,t}} G \\
&\quad = \operatorname{Re}\left(h/\tilde{h}\right)\psi'(t)\left\langle B_{z,\psi(t)}(0,0)X, \left(\frac{X_{\text{hyp}}}{1+|X|^2_{\text{hyp}}}, iX_{\text{ell}}\right)\right\rangle \\
&\qquad\qquad \cdot\left(1+\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}(|\Xi|)\right) \\
&\qquad + \operatorname{Re}\left(h/\tilde{h}\right)\psi'(t)\left\langle B_{z,\psi(t)}(0,0)\Xi, \left(\frac{\Xi_{\text{hyp}}}{1+|\Xi|^2_{\text{hyp}}}, i\Xi_{\text{ell}}\right)\right\rangle \\
&\qquad\qquad \cdot\left(1+\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}(|X|)\right).
\end{aligned}$$

Now since $B_{z,\psi(t)}(0,0)$ is block diagonal of the form (6.14), [Chr1, Theorem 4] yields a linear symplectomorphism κ_1 such that

$$\begin{aligned}
&\operatorname{Re}\kappa_1^*(H_{q^2_{z,t}}(G)) = \\
&\quad = \left(h/\tilde{h}\right)\psi'(t)\left[\frac{\sum_{j=1}^{n-n_e-1} r_j^{-2}X_j^2}{1+|MX|^2}\left(1+\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}(|\Xi|)\right)\right. \\
&\qquad \left.+\frac{\sum_{j=1}^{n-n_e-1} r_j^{-2}\Xi_j^2}{1+|M'\Xi|^2}\left(1+\left(h/\tilde{h}\right)^{\frac{1}{2}}\mathcal{O}(|X|)\right)\right],
\end{aligned}$$

where M and M' are nonsingular. Thus, since $\chi(t)\psi_1'(t) = 0$,

$$\begin{aligned}
&\operatorname{Im}\widetilde{Q}_{z,t} \\
&\quad = \operatorname{Im} s\chi(t)[Q^2_{z,t}, G^w] + s\chi(t)E_1^w + s^2\chi(t)^2E_2^w \\
&\quad = -sh\chi(t)(A_1(1+E_0) + A_2(1+E_0'))^w \\
&\qquad + s\chi(t)E_1^w + s^2\chi(t)^2E_2^w,
\end{aligned} \tag{7.21}$$

with $E_0, E_0' = \mathcal{O}(\epsilon)$, $E_1 = \mathcal{O}(h^{3/2}\tilde{h}^{3/2})$, $E_2 = \mathcal{O}(h\tilde{h})$, and $(A_1 + A_2)^w =: A^w = \operatorname{Op}_{\tilde{h}}^w(A)$ for

$$A(X,\Xi) = \psi'(t)(\kappa_1^{-1})^*\left(\frac{\sum_{j=1}^{n-n_e-1} r_j^{-2}X_j^2}{1+|MX|^2} + \frac{\sum_{j=1}^{n-n_e-1} r_j^{-2}\Xi_j^2}{1+|M'\Xi|^2}\right). \tag{7.22}$$

From [EvZw, Theorem 10.3] there is a unitary h-FIO F_1 quantizing κ_1^{-1} so that

$$\tilde{A} := F_1\operatorname{Op}_{\tilde{h}}^w(A)F_1^{-1} = \operatorname{Op}_{\tilde{h}}^w(\kappa_1^*A) + \mathcal{O}(\tilde{h}^2).$$

We claim that for $\tilde{h}$ sufficiently small and $\tilde{v}$ smooth,

$$\langle \tilde{A}^w\tilde{v}, \tilde{v}\rangle \geq \frac{\tilde{h}}{C}\|\tilde{v}\|^2$$

for some constant $C > 0$, which is essentially the lower bound for the harmonic oscillator $\tilde{h}^2D_X^2 + X^2$. It suffices to prove this inequality for individual j, which is the content of [Chr1, Lemma 5.1]. As F_1 is unitary, setting $\tilde{v} = F_1\tilde{u}$ for $\tilde{u}$ smooth gives

$$
\begin{aligned}
\langle A^w \tilde{u}, \tilde{u} \rangle &\geq \frac{\tilde{h}}{C} \|\tilde{u}\|^2 - \mathcal{O}(\tilde{h}^2)\|\tilde{u}\|^2 \\
&\geq \frac{\tilde{h}}{C'} \|\tilde{u}\|^2, \qquad (7.23)
\end{aligned}
$$

for $\tilde{h} > 0$ sufficiently small.

Now fix $\tilde{h} > 0$ and $|s| > 0$ sufficiently small so that the estimate (7.23) holds and the errors E_1 and E_2 satisfy

$$
\|shA^w \tilde{u}\|_{L^2} \gg \|sE_1^w \tilde{u}\|_{L^2} + \|s^2 E_2^w \tilde{u}\|_{L^2},
$$

and fix $\epsilon > 0$ sufficiently small that the errors $|E_0|, |E_0|' \ll 1$, independent of $h > 0$.

For $\tilde{u}$ a smooth function with wavefront set contained in U, we now have

$$
\begin{aligned}
-\operatorname{Im} \langle \widetilde{Q}_{z,t} \tilde{u}, \tilde{u} \rangle &\geq \psi'(t)\chi(t) \frac{h\tilde{h}}{C} \|\tilde{u}\|^2 \\
&= \psi'(t) \frac{h\tilde{h}}{C} \|\tilde{u}\|^2,
\end{aligned}
$$

since $\chi(t) \equiv 1$ on the support of $\psi_1'(t)$. This is (7.11), the crucial estimate needed for Step 4.

If $\tilde{q}_{z,t}$ is not in the form (6.13), by Proposition 6.3 there is a symplectomorphism κ_2 so that $\kappa_2^* \tilde{q}_{z,t}$ is of the form (6.13). Using [EvZw, Theorem 10.3] to quantize κ_2 as an h-FIO F_2, we get

$$
\operatorname{Op}_h^w (\kappa^* \tilde{q}_{z,t} + E_1) = F^{-1} \widetilde{Q}_{z,t} F,
$$

where $E_1 = \mathcal{O}(h^2)$ is the error arising from [EvZw, Theorem 10.3]. We may then use the previous argument for $\kappa_2^* q_{z,t}$ getting an additional error of $\mathcal{O}(h^2)$ from [EvZw, Theorem 10.3] in (7.11).

7.5. **Step 4: Estimation of $\widetilde{W}$.** Let $v \in L^2(V)$ with wavefront set sufficiently close to $(0,0)$, and set $\tilde{v} = T_{h,\tilde{h}} v$. Now $\widetilde{W}(t)$ is no longer unitary, so we calculate

$$
\begin{aligned}
\partial_t \left\langle \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle &= 2 \left\langle \partial_t \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle \\
&= \frac{2i}{h} \left\langle \left(\widetilde{Q}_{z,t} + \mathcal{O}(h^{3/2}\tilde{h}^{3/2}) \right) \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle \\
&= -\frac{2}{h} \left\langle \left(\operatorname{Im} \widetilde{Q}_{z,t} + \mathcal{O}(h^{3/2}\tilde{h}^{3/2}) \right) \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle \\
&\geq C^{-1} \left(\psi'(t)\tilde{h} - \mathcal{O}(h^{1/2}\tilde{h}^{3/2}) \right) \left\langle \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle.
\end{aligned}
$$

Thus there is a positive constant C such that

$$
\partial_t \left(\left\langle \widetilde{W}(t)\tilde{v}, \widetilde{W}(t)\tilde{v} \right\rangle e^{-(\psi(t)\tilde{h} - \mathcal{O}(h^{1/2}\tilde{h}^{3/2}))/C} \right) \geq 0,
$$

so

$$
\left\| \widetilde{W}(t)\tilde{v} \right\|^2 \geq e^{\psi(t)(\tilde{h} - \mathcal{O}(h^{1/2}\tilde{h}^{3/2}))/C} \|\widetilde{W}(0)\tilde{v}\|^2
$$

and since $\psi(1) = 1$, shrinking $\tilde{h} > 0$ if necessary, we have for $0 < h \leq h_0$ sufficiently small,

$$
\left\| \widetilde{W}(1)\tilde{v} \right\| \geq R \|\widetilde{W}(0)\tilde{v}\|, \quad R > 1 \text{ independent of } 0 < h \leq h_0.
$$

Now

$$\begin{aligned}\widetilde{W}(0) &= e^{-s\chi(0)G^w} T_{h,\tilde{h}} W^z(0) T_{h,\tilde{h}}^{-1} e^{s\chi(0)G^w} \\ &= T_{h,\tilde{h}} W^z(0) T_{h,\tilde{h}}^{-1}\end{aligned}$$

is unitary, so

$$\left\| \widetilde{W}(1)\tilde{v} \right\| \geq R\|\tilde{v}\|, \tag{7.24}$$

independent of $0 < h \leq h_0$.

As in §5, let the operators K^w be defined by

$$e^{sK^w} = T_{h,\tilde{h}}^{-1} e^{s\chi(1)G^w} T_{h,\tilde{h}} = T_{h,\tilde{h}}^{-1} e^{sG^w} T_{h,\tilde{h}},$$

so that

$$\widetilde{W}(1) = e^{-sK^w} M(z)^{-1} e^{sK^w},$$

and Theorem 6 is proved. □

Remark 7.2. The error arising at the end of the proof of Theorem 6 from the use of [EvZw, Theorem 10.3] is of order $\mathcal{O}(h^2)$ and hence negligible compared to our lower bound of h for A. However, the estimate of A is used for the imaginary part of $\widetilde{Q}_{z,t}$, and the error in [EvZw, Theorem 10.3] is real, so $\mathcal{O}(h)$ would have been sufficient. This means the analysis above does not strictly depend on using the Weyl calculus.

Remark 7.3. It is interesting to note that the estimate (1.3) depends only on the real parts of the eigenvalues λ_j above. Unraveling the definitions, the eigenvalues λ_j are logarithms of the eigenvalues of the linearized Poincaré map $dS(0)$ from above. Then (1.3) depends only on the modulis of the eigenvalues of $dS(0)$ which lie off the unit circle. We interpret this as a quantum analogue of the fact that $dS(0,0)$ is semi-hyperbolic.

8. Proof of the Main Theorems

The Main Theorem follows exactly as the Main Theorem in [Chr1] with the corrections in [Chr1a]. The proofs of Theorems 1' and 2' procede with very little modification. The only things left to do are to prove Theorems 3 and 3′ and indicate how to prove Main Theorem'.

8.1. **Proof of Theorems 3 and** $3'$**.** Owing to [BuZw, Lemma A.2], we only need to prove that in both cases $\widetilde{Q}(z)$ satisfies a polynomial estimate of the form

$$\|\widetilde{Q}(z)^{-1}\|_{L^2(X)\to L^2(X)} \leq Ch^N \tag{8.1}$$

for some N and $z \in [-\epsilon, \epsilon] + i(-c_0 h, c_0 h)$. The operator $P(h) - z$ satisfies a similar estimate *microlocally* near γ, so we have to glue the microlocal estimate into the better propagation estimates. We first observe that for any $\delta > 0$ we can take c_0 sufficiently small and interpolate to get

$$\|P(-z)u\| \geq h^{1+\delta}/C\|u\|$$

for u with wavefront set sufficiently close to γ.

Choose $\psi_0 \in \mathcal{C}^\infty(T^*X)$ satisfying $\psi_0 \equiv 1$ near γ with small support. Choose $W \in \mathcal{C}^\infty(T^*X)$ so that $W \equiv 1$ away from γ and $W\psi_0 = 0$. Finally, fix $\epsilon > 0$ and choose $\psi_1 \in \mathcal{C}_c^\infty(T^*X)$ so that $W \geq \epsilon > 0$ on $\operatorname{supp}(1-\psi_1)$. Then

$$\begin{aligned} \|\psi_0 u\| &\leq Ch^{-1-\delta}\|(P-z)\psi_0 u\| + \mathcal{O}(h^\infty)\|u\| \\ &\leq Ch^{-1-\delta}\|\widetilde{Q}(z)u\| + Ch^{-\delta}\|\tilde{\psi}u\| + \mathcal{O}(h^\infty)\|u\| \end{aligned}$$

where $\tilde{\psi} \equiv 1$ on $\mathrm{WF}_h[P, \psi_0]$. But then the propagation estimate [Chr2, Lemma 2.4] (trivially modified to the complex case) implies

$$\|\tilde{\psi}u\| \leq Ch^{-1}\|\widetilde{Q}(z)u\| + C\|(1-\psi_1)u\|.$$

But $|\operatorname{Im} z| \leq c_0 h$ implies $W + \operatorname{Im} z \geq \epsilon - c_0 h \geq \epsilon/2$ on $\operatorname{supp}(1-\psi_1)$. Hence

$$\begin{aligned} \|(1-\psi_1)u\|^2 &\leq C\left\langle (W + \operatorname{Im} z)(1-\psi_1)u, (1-\psi_1)u\right\rangle \\ &= -C\operatorname{Im}\left\langle \widetilde{Q}(z)(1-\psi_1)u, (1-\psi_1)u\right\rangle \\ &= -\operatorname{Im}\left\langle (1-\psi_1)^*(1-\psi_1)\widetilde{Q}(z)u, u\right\rangle + \mathcal{O}(h)\|u\|^2 \\ &\leq Ch^{-1}\|\widetilde{Q}(z)u\|^2 + Ch^1\|u\|^2. \end{aligned}$$

Similarly, we use propagation again to estimate

$$\|(1-\psi_0)u\|^2 \leq Ch^{-1}\|\widetilde{Q}(z)u\|^2 + C\|(1-\psi_1)u\|^2 + \mathcal{O}(h^\infty)\|u\|^2,$$

so that

$$\begin{aligned} \|u\| &\leq \|\psi_0 u\| + \|(1-\psi_0)u\| \\ &\leq Ch^{-1-\delta}\|\widetilde{Q}(z)u\| + Ch^{1/2-\delta}\|u\|^2. \end{aligned}$$

Taking $\delta < 1/2$ yields (8.1).

8.2. **Proof of the Main Theorem.** Let W^w be a symbol which is microlocally 1 away from γ, and for $z \in [-\epsilon_0, \epsilon_0] + i(-c_0 h, c_0 h)$, define as in (1.6)

$$\widetilde{Q}(z) := P(h) - z - iW^w. \tag{8.2}$$

For the analysis near the boundary, choose also $\psi \in \mathcal{C}^\infty(\mathbb{R})$ satisfying (2.6-2.7).

Let $m_j^\pm \in T^*X$, for $j = 1, \ldots, K$ denote the points where γ reflects off the boundary, with $m_j^\pm$ denoting the point of intersection with the boundary of the outgoing and incoming bicharacteristics respectively, and let m_j be the projection of $m_j^\pm$ onto $T^*(\partial X)$. Let $U_j \subset T^*(\partial X)$ denote a neighbourhood of m_j which is small enough so that a factorization of P as in Lemma 2.1 is possible in a neighbourhood of U_j. Shrinking U_j if necessary, we assume also that the construction in Lemma 2.6 is valid in a neighbourhood of U_j. That is, if P is factorized as in Lemma 2.1 near U_j, we write

$$P = (hD_1 - A_-(x, hD'))(hD_1 - A_+(x, hD')) \text{ near } m_j,$$

and there is an operator $A_{b,j}(x, hD')$ which is 1 microlocally near U_j, zero away from U_j and commutes with $(hD_1 - A_+(x, hD'))$ microlocally near U_j.

Let $\gamma_\pm^j$ be a small interval on the outgoing/incoming bicharacteristic near $m_j^\pm$, and let $\widetilde{U}_j \subset T^*X$ be a neighbourhood of $\gamma_\pm^j$ such that

$$(\mathrm{WF}_h(\psi(P)A_{b,j}))|_{\partial X} \subset \widetilde{U}_j,$$

and $\psi(P)A_{b,j} \equiv 1$ on $\gamma_\pm^j \cap \widetilde{U}_j$. Choose $\chi_j \in \mathcal{C}_c^\infty(T^*X)$, $\chi_j \equiv 1$ on $\widetilde{U}_j$ with sufficiently small support that

$$\psi(P)A_{b,j} \equiv 1 \text{ on } \operatorname{supp} \nabla \chi_j \cap \gamma. \tag{8.3}$$

Finally, set

$$\chi_0 = 1 - \sum_j \chi_j.$$

Now for $A \in \Psi_{h,db}^{0,0}$ as in the statement of the Main Theorem' with wavefront set sufficiently close to γ, let $A_0 \in \Psi_h^{0,0}$ have wavefront set close to γ and satisfy

$$\begin{aligned} & A_0 \equiv 1 \text{ on } \left\{ \mathrm{WF}_h A \setminus \left\{ \bigcup_j \widetilde{U}_j \right\} \right\}, \\ & A_0 \equiv 1 \text{ on } \left(\bigcup_{j=0}^K \operatorname{supp} \nabla \chi_j \right) \cap \gamma, \qquad (8.4) \\ & A_0 \equiv 0 \text{ elsewhere }. \end{aligned}$$

We define $\tilde{A} \in \Psi_{h,db}^{0,0}$ satisfying

$$\tilde{A} \equiv 1 \text{ on } \mathrm{WF}_h A \tag{8.5}$$

by

$$\tilde{A} = \chi_0 A_0 + \sum_j \chi_j \psi(P) A_{b,j},$$

where ψ satisfies (2.6-2.7). Observe if $\mathrm{WF}_h A$ is sufficiently close to γ, $\tilde{A}$ satisfies (8.5). We have $Q(0)\tilde{A}u = P(h)\tilde{A}u$ since $\mathrm{WF}_h a^w \cap \mathrm{WF}_h \tilde{A} = \emptyset$. But

$$P(h)\tilde{A}u = \left[P(h), \tilde{A} \right] u + \tilde{A}P(h)u \tag{8.6}$$

and we claim

$$\begin{aligned} \left\| \left[P, \tilde{A} \right] u \right\| &= \left\| [P, \chi_0 A_0]\, u + \sum_j \left([P, \chi_j \psi(P) A_{b,j}] \right) u \right\| \\ &= \mathcal{O}(h) \left\| (I - A)u \right\|. \qquad (8.7) \end{aligned}$$

To see this, we observe for $u \in \mathcal{C}^\infty(X) \cap L^2(X)$,

$$\begin{aligned} & [P, \chi_0 A_0]\, u + \sum_j [P, \chi_j A_{b.j}] = \\ &= \chi_0 [P, A_0]\, u + [P, \chi_0] A_0 u \\ &\quad + \sum_j \left(\chi_j \psi(P) [P, A_{b,j}] + [P, \chi_j]\, \psi(P) A_{b,j} \right) u \end{aligned}$$

We have

$$\|([P, \chi_0] A_0 + \sum_j [P, \chi_j]\, \psi(P) A_{b,j}) u\| \leq Ch \|(I - A)u\|$$

from (8.3) and (8.4). These two conditions also imply

$$\mathrm{WF}_h \chi_0 \, [P, A_0] \cap \gamma = \emptyset \text{ and}$$
$$\mathrm{WF}_h \chi_j \psi(P) \, [P, A_{b,j}] \cap \gamma = \emptyset,$$

and the symbol of A_0 is compactly suppported away from the boundary, so

$$\|\chi_0 \, [P, A_0] \, u\| \leq Ch\|(I - A)u\|.$$

For each j, it suffices to consider the remaining terms in local coordinates at the boundary. Fix j and assume we are in the coordinates used in Lemma 2.6 in U_j:

$$\begin{aligned} &\chi_j \psi(P) \, [P, A_{b,j}(x, hD')] = \\ &\quad = \quad \chi_j \psi(P) \, [(hD_1 - A_-(x, hD'))(hD_1 - A_+(x, hD')), A_{b,j}] \\ &\quad = \quad \chi_j \psi(P) \, [(hD_1 - A_-(x, hD')), A_{b,j}(x, hD')] \, (hD_1 - A_+(x, hD')), \end{aligned}$$

since $A_{b,j}$ commutes with $(hD_1 - A_+(x, hD'))$. The principal symbol of

$$\chi_j \psi(P) \, [(hD_1 - A_-(x, hD')), A_{b,j}(x, hD')]$$

is

$$\frac{h}{i} \chi_j \psi((\xi_1 - r^{\frac{1}{2}}(x, \xi'))(\xi_1 + r^{\frac{1}{2}}(x, \xi'))) \left\{ (\xi_1 + r^{\frac{1}{2}}(x, \xi')), \sigma_h(A_{b,j}^+)(x, \xi') \right\},$$

which is $\mathcal{O}(h)$ and has h-wavefront set away from γ. Summing over j gives (8.7).

We now use the control theory arguments from [BuZw] and [Chr1a]. That is, for $z \in [-1/2, 1/2]$

$$\begin{aligned} \|\tilde{A}u\| \leq & C\|\widetilde{Q}(z)^{-1}\widetilde{Q}(z)\tilde{A}u\| + \mathcal{O}(h^\infty)\|u\| \\ = & C\|\widetilde{Q}(z)^{-1}(\tilde{A}(P(h) - z) + [P(h), \tilde{A}])u\| + \mathcal{O}(h^\infty)\|u\| \\ \leq & C\|\widetilde{Q}(z)^{-1}\tilde{A}(P(h) - z)u\| + C\|\widetilde{Q}(z)^{-1}\varphi[P(h), \tilde{A}]u\| + \mathcal{O}(h^\infty)\|u\| \\ \leq & C\frac{\log(1/h)}{h}\|(P(h) - z)u\| + C\log^{1/2}(1/h)\|(I - A)u\| + \mathcal{O}(h^\infty)\|u\|. \end{aligned} \tag{8.8}$$

Here we have used that $\widetilde{Q}(z)A = (P(h) - z)A$, (8.6), (2.5) and $P(h) - z$ is elliptic away from $\{p = z\} \supset \gamma$.

This gives

$$\begin{aligned} \|u\| \quad \leq & \quad \|\tilde{A}u\| + \|(I - A)u\| \\ \leq & \quad C\frac{\log(1/h)}{h}\|(P(h) - z)u\| + C(\log^{1/2}(1/h) + 1)\|(I - A)u\| + \mathcal{O}(h^\infty)\|u\| \end{aligned}$$

which proves the Main Theorem'.

8.3. **The proof of Theorem 2 and Theorem $2'$.** In order to prove Theorem 2 and Theorem $2'$, which indicate the complex absorption term need only be of size h. We first repeat the calculations leading to (8.8) with $P(h)$ replaced by $Q(z) = P(h) - z - iha^w$

Next, we assume $a^w = B^*B$ for some non-negative definite $B \in \Psi^0$, so that

$$\|Bu\|^2 = \langle au, u \rangle,$$

and [Chr2, Lemma 2.4] implies

$$\|(1 - A)u\| \leq \frac{C}{h}\|Q(z)u\| + C\|Bu\| + \mathcal{O}(h^\infty)\|u\|.$$

Then, again using [Chr2, Lemma 2.4] on the term with $(1-A)u$ in (8.8) all told we have the estimate

$$\|u\| \leq C\frac{\log(1/h)}{h}\|Q(z)u\| + C\log^{1/2}(1/h)\|Au\| + \mathcal{O}(h^\infty)\|u\|.$$

Finally, to get the estimates (1.5) and (2.3) (and the improvement to a complex neighbourhood), we calculate for $z \in [-1/2, 1/2]$,

$$\begin{aligned}\|Au\|^2 &= \langle au, u\rangle \\ &= \frac{1}{h}\,\mathrm{Im}\,\langle Q(z)u, u\rangle \\ &\leq \frac{1}{h}\|Q(z)u\|\|u\|,\end{aligned}$$

so that we have for any $\epsilon > 0$,

$$\begin{aligned}\log^{1/2}(1/h)\|Au\| &\leq \log^{1/2}(1/h)\left(\frac{1}{h}\|Q(z)u\|\|u\|\right)^{1/2} \\ &\leq \frac{\log(1/h)}{2\epsilon h}\|Q(z)u\| + \epsilon\|u\|.\end{aligned}$$

Combining this with (8.8) and taking $\epsilon > 0$ sufficiently small yields Theorems 2 and $2'$. The improvement to $|\,\mathrm{Im}\,z| \leq ch/\log(1/h)$ follows from taking $c > 0$ sufficiently small, since then the order of the perturbation is the same order as the estimate.

9. An Application: Quasimodes near Elliptic Orbits

In this section, we show how the techniques of reducing microlocal estimates near a periodic orbit to estimates on an h-Fourier integral operator acting microlocally on the Poincaré section via the *Quantum Monodromy operator* from [SjZw1] and §4 can be used with the quasimode construction in [ISZ] to produce well-localized quasimodes near an elliptic periodic orbit. We also give estimates on the number and location of approximate eigenvalues associated to the quasimodes.

Let X be a smooth, compact manifold, $\dim X = n$, and suppose $P \in \Psi^{k,0}(X)$, $k \geq 1$, be a semiclassical pseudodifferential operator of real principal type which is semiclassically elliptic outside a compact subset of T^*X as in the introduction. Let $\Phi_t = \exp tH_p$ be the classical flow of p and assume there is a closed *elliptic* orbit $\gamma \subset \{p = 0\}$. That γ is elliptic means if $N \subset \{p = 0\}$ is a Poincar'e section for γ and $S : N \to S(N)$ is the Poincaré map, then $dS(0,0)$ has eigenvalues all of modulus 1. We will also need the following non-resonance assumption:

$$\left\{\begin{array}{l}\text{if } e^{\pm i\alpha_1}, e^{\pm i\alpha_2}, \ldots, e^{\pm i\alpha_k} \text{ are eigenvalues of } dS(0,0), \text{ then} \\ \alpha_1, \alpha_2, \ldots, \alpha_k \text{ are independent over } \pi\mathbb{Z}.\end{array}\right. \tag{9.1}$$

Finally, we assume if $\gamma \cap \partial X \neq \emptyset$ then γ reflects only transversally off ∂X, ∂X is noncharacteristic with respect to P, and $P \in \mathrm{Diff}^{2,0}_{h,db}$.

Under these assumptions, it is well known that there is a family of elliptic closed orbits $\gamma_z \subset \{p = z\}$ for z near 0, with $\gamma_0 = \gamma$. In this work we consider the following eigenvalue problem for z in a neighbourhood of $z = 0$:

$$\left\{\begin{array}{l}(P - z)u = 0; \\ \|u\|_{L^2(X)} = 1.\end{array}\right. \tag{9.2}$$

We prove the following Theorem.

Theorem 7. *For each $m \in \mathbb{Z}$, $m > 1$, and each $c_0 > 0$ sufficiently small, there is a finite, distinct family of values*

$$\{z_j\}_{j=1}^{N(h)} \subset [-c_0 h^{1/m}, c_0 h^{1/m}]$$

and a family of quasimodes $\{u_j\} = \{u_j(h)\}$ with

$$\mathrm{WF}_h u_j = \gamma_{z_j},$$

satisfying

$$\left\{ \begin{array}{l} (P - z_j)u_j = \mathcal{O}(h^\infty)\|u_j\|_{L^2(X)}; \\ \|u_j\|_{L^2(X)} = 1. \end{array} \right. \tag{9.3}$$

Further, for each $m \in \mathbb{Z}$, $m > 1$, there is a constant $C = C(c_0, 1/m)$ such that

$$C^{-1} h^{-n(1-1/m)} \leq N(h) \leq C h^{-n}. \tag{9.4}$$

9.1. **The Model Case.** We consider the case $n = 2$, the first nontrivial dimension. Recall the model for p near an elliptic periodic orbit is $p \in \mathcal{C}^\infty(T^*(\mathbb{S}^1 \times \mathbb{R}))$,

$$p = \tau + \frac{\alpha}{2}(x^2 + \xi^2),$$

with $\alpha > 0$ satisfying $\alpha \notin \pi\mathbb{Z}$. Then we study (9.2) for

$$P = hD_t + \frac{\alpha}{2}(x^2 + h^2 D_x^2).$$

Let

$$\begin{aligned} Q &= \frac{\alpha}{2}(x^2 + h^2 D_x^2) \\ &= \mathrm{Op}_h^w \left(\frac{\alpha}{2}(x^2 + \xi^2) \right). \end{aligned}$$

Q is just $\alpha/2$ times the harmonic osciallator, so we have

$$Q v_k = h \frac{\alpha}{2}(2k+1) v_k$$

for

$$\begin{aligned} v_k &:= h^{-1/4} H_k(x/h^{\frac{1}{2}}) e^{-x^2/2h}, \\ \|v_k\|_{L^2} &= 1, \end{aligned}$$

where H_k are the (normalized) Hermite polynomials of degree k (see, for example, [EvZw]). Note $\mathrm{WF}_h v_k = (0,0)$. Now we make an ansatz of

$$u = g_k(t) v_k(x)$$

for $g_k(t)$ to be determined. Plugging u into (9.2) yields

$$hD_t g_k + \frac{\alpha}{2} h(2k+1) g_k = z g_k,$$

which implies

$$g_k(t) = \exp\left(\frac{it}{h} \left(z - \frac{\alpha}{2}(2k+1)h \right) \right).$$

Since the spectrum of hD_t on $\mathbb{S}^1$ is $\{2\pi m h\}_{m \in \mathbb{Z}}$, we have

$$z = \frac{\alpha}{2}(2k+1)h + 2\pi m h. \tag{9.5}$$

In the model case, since there is no microlocalization necessary (and, in particular, p is not elliptic at infinity), we actually have dense spectrum in any interval.

In order to motivate our general construction, we present the same example from the point of view of the monodromy operator. Here we think of $Q-z$ as a z-dependent family of operators on $L^2(V)$, where $V \subset \mathbb{R}$ is an open neighbourhood of 0. Then the monodromy operator $M(z)$ is defined microlocally as the time $t=1$ solution to the ordinary differential equation

$$\begin{cases} hD_t M(z,t) + (Q-z)M(z,t) = 0, \\ M(z,0) = \mathrm{id}_{L^2(V)\to L^2(V)}. \end{cases} \tag{9.6}$$

Our general technique will be to find eigenfunctions of $M(z) = M(z,1)$ with eigenvalue 1. Using again v_k as in the previous paragraph, we try

$$M(z,t)v_k = e^{-i2\pi mt}v_k,$$

with $m \in \mathbb{Z}$ so that $M(z,1)v_k = v_k$. This yields from (9.6)

$$\left(-h2\pi m + \frac{\alpha}{2}h(2k+1) - z\right)v_k = 0$$

which is the same as (9.5).

9.2. **Quasimodes on the Poincaré section.** Theorem 5 and the definition of the monodromy operator $M(z)$ motivate us to study the normal form for a family of elliptic symplectomorphisms

$$S_z : W_1 \to W_2$$

under the nonresonance condition (9.1) on $dS(0)$, where W_1 and W_2 are neighbourhoods of $0 \in \mathbb{R}^{2n-2}$. We use the standard notation of [ISZ] and write

$$\begin{aligned} \imath_j &= x_j^2 + \xi_j^2, \text{ and} \\ I_j &= \imath_j^w = x_j^2 + h^2 D_{x_j}^2. \end{aligned}$$

According to the results of [IaSj] and [ISZ], there is a symplectic choice of coordinates near $(x,\xi;z) = (0,0;0)$ such that

$$S_z = \exp H_{q_z} + \mathcal{O}((x,\xi;z)^\infty), \tag{9.7}$$

for

$$q_z = \sum_{j=1}^{n-1} \lambda_j(z)\imath_j + R(z,\imath_1,\ldots,\imath_{n-1}).$$

Here the remainder $R(z,\imath) = \mathcal{O}(\imath^2)$ and the $\lambda_j(z)$ are positive and depend smoothly on z.

Further, if $M(z)$ is the monodromy operator quantizing S_z and

$$\text{(i) } z \in [-\epsilon_0 h^{1/m}, \epsilon_0 h^{1/m}] + i(-c_0 h, c_0 h), \tag{9.8}$$

$$\text{(ii) } \imath_j \leq h^{1/m} \tag{9.9}$$

for $m \in \mathbb{Z}$, $m > 1$, then there is a family of unitary h-FIOs $V(z)$ such that

$$e^{iz/h}M(z) = V(z)^{-1}e^{-i(Q(z,h)-z)/h}V(z) + \mathcal{O}_{L^2\to L^2}(h^\infty), \tag{9.10}$$

where

$$\begin{aligned} Q(z,h) &= \sum_{j=0}^{\infty} h^j q_j(z,I), \text{ with} \\ q_j(z,I) &= \mathcal{O}(I) \end{aligned} \tag{9.11}$$

and

$$q_0(z, \imath) = q_z(\imath).$$

Now let $\beta \in \mathbb{N}^{n-1}$ be a multi-index and define

$$v_\beta = c_\beta h^{-(n-1)/4} e^{-|x|^2/2h} \prod_{j=1}^{n-1} H_{\beta_j}(x_j/h^{\frac{1}{2}}),$$

with H_{β_j} the Hermite polynomials as in §9.1 and c_β chosen independent of h to normalize v_β in $L^2(\mathbb{R}^{n-1})$. The functions v_β satisfy

$$I_j v_\beta = h(2\beta_j + 1)v_\beta,$$

and with $\mathbb{1} = (1, \ldots, 1) \in \mathbb{N}^{n-1}$ we write

$$I v_\beta = h(2\beta + \mathbb{1})v_\beta.$$

Hence we have

$$\begin{aligned} Q(z,h)v_\beta &= \left(\sum_{j=0}^{\infty} h^j q_j(z, h(2\beta + \mathbb{1})) \right) v_\beta \\ &=: \zeta_\beta(z) v_\beta, \qquad (9.12) \end{aligned}$$

where

$$\zeta_\beta(z) = h \sum_{j=1}^{n-1} \lambda_j(z)(2\beta_j + 1) + \mathcal{O}(h^2).$$

The quantization condition (9.9) implies we have the restriction on ζ_β:

$$|h \sum_{j=1}^{n-1} \lambda_j(z)(2\beta_j + 1)| \leq C h^{1/m},$$

for $0 < 1/m < 1$, giving

$$\begin{aligned} \#\{\zeta_\beta(z)\} &= \# \left\{ \left| h \sum_{j=1}^{n-1} \lambda_j(z)(2\beta_j + 1) \right| \leq C h^{1/m} \right\} \\ &\simeq \#\{|\beta| \leq h^{1/m-1}\} \\ &\simeq h^{(1/m-1)(n-1)} + o(1). \qquad (9.13) \end{aligned}$$

9.3. **The proof of Theorem 7.** Observe the functions v_β constructed above satisfy

$$\mathrm{WF}_h v_\beta = (0,0) \in \mathbb{R}^{2n-2}.$$

Beginning with v_β we want to construct $\tilde{v}_{\beta,k}$ and find values of z, β, and $k \in \mathbb{Z}$ so that

$$(\mathrm{\ id\ } - M(z))\tilde{v}_\beta = \mathcal{O}(h^\infty).$$

Let

$$\widetilde{M}(z) = V(z)M(z)V(z)^{-1} = e^{-i(Q(z,h)-z)/h}$$

with $V(z)$ and $Q(z,h)$ as in (9.10), and observe $\widetilde{M}(z) = \widetilde{M}(z, 1)$ for

$$\widetilde{M}(z,t) = \exp(-it(Q(z,h) - z)/h)$$

satisfying

$$\begin{cases} hD_t\widetilde{M}(z,t) + Q(z,h)\widetilde{M}(z,t) = z\widetilde{M}(z,t) \\ \widetilde{M}(z,0) = \text{id}\,. \end{cases} \tag{9.14}$$

The spectrum of hD_t on $\mathbb{R}/\mathbb{Z}$ is $\{h2\pi k\}$ for $k \in \mathbb{Z}$, so we want the solution space to (9.14) intersected with the solution space to

$$(e^{iz/h} - \widetilde{M}(z,1))v = v$$

to contain the "ansatz" space

$$v_{k,\beta}(t,x) := e^{-it2\pi k} v_\beta(x). \tag{9.15}$$

More precisely, $v_{k,\beta}(1,x) = v_\beta x$, so we want to solve

$$\begin{cases} hD_t\widetilde{M}(z,t)v_{\beta,k} + (Q(z,h) - z)\widetilde{M}(z,t)v_{\beta,k} = -z\widetilde{M}(z,t)v_{\beta,k} \\ \widetilde{M}(z,0)v_{\beta,k} = v_{\beta,k}. \end{cases}$$

That is, we want to find z satisfying

$$2z - \zeta_\beta(z) = 2\pi kh,$$

where $\zeta_\beta(z)$ is given by (9.12).

Expanding $Q(z,h)$ in a formal series in z as we may do according to the quantization condition (9.9), we write

$$Q(z,h) = \sum_{l=0}^{\infty} z^l Q_l(h,I) \tag{9.16}$$

microlocally, with

$$Q_0 = \sum_{j=1}^{n-1} \lambda_j(0) I_j + \mathcal{O}(I^2),$$

and

$$Q_l = \mathcal{O}(I).$$

Hence we will seek

$$z_{k,\beta} = \sum_{j=0}^{\infty} z_{k,\beta}^{(j)}, \tag{9.17}$$

with $z_{k,\beta}^{(j)} = \mathcal{O}(h^{(j+1)/m})$. For $z_{k,\beta}^{(0)}$, we solve

$$2z_{k,\beta}^{(0)} = h\sum_{j=1}^{n-1} \lambda_j(0)(2\beta_j + 1) + 2k\pi h$$

which is $\mathcal{O}(h^{1/m})$ if

$$|k| \leq Ch^{1/m-1}. \tag{9.18}$$

For $z_{k,\beta}^{(1)}$ we plug $z_{k,\beta}^{(0)} + z_{k,\beta}^{(1)}$ into (9.16) to get the equation

$$\begin{aligned} 2z_{k,\beta}^{(0)} + 2z_{k,\beta}^{(1)} &= h\sum_{j=1}^{n-1} \lambda_j(0)(2\beta_j + 1) + 2k\pi h + \sum_{l=1}^{\infty} (z_{k,\beta}^{(0)} + z_{k,\beta}^{(1)})^l Q_l(h, h(2\beta + \mathbb{1})) \\ &= 2z_{k,\beta}^{(0)} + z_{k,\beta}^{(0)} Q_l(h, h(2\beta + \mathbb{1})) + \mathcal{O}(h^{3/m}), \end{aligned}$$

provided $z_{k,\beta}^{(1)} = \mathcal{O}(h^{2/m})$. Hence we choose

$$2z_{k,\beta}^{(1)} = z_{k,\beta}^{(0)} Q_l(h, h(2\beta + \mathbb{1})).$$

Continuing in this fashion, we select $z_{k,\beta}^{(j)}$ for $j \geq 2$ using the following equation:

$$2\sum_{r=0}^{j} z_{k,\beta}^{(r)} = \sum_{r=0}^{j-1} \left(\sum_{l=0}^{j-r-1} z_{k,\beta}^{(l)} \right)^r Q_r(h, h(2\beta + \mathbb{1})),$$

modulo $\mathcal{O}(h^{(j+2)/m})$, hence $z_{k,\beta}^{(j)} = \mathcal{O}(h^{(j+1)/m})$.

Now there is no reason why (9.17) should converge in any sense, so we want to find a convergent series

$$\tilde{z}_{k,\beta} = \sum_{j=0}^{\infty} \tilde{z}_{k,\beta}^{(j)}$$

with $\tilde{z}_{k,\beta}^{(j)} = \mathcal{O}(h^{(j+1)/m})$, satisfying

$$\tilde{z}_{k,\beta} - \sum_{0}^{mN} z_{k,\beta}^{(j)} = \mathcal{O}(h^N) \tag{9.19}$$

for every $N > 0$. For this, we follow the proof of Borel's Lemma from [EvZw]. Choose $\chi \in \mathcal{C}_c^\infty([-1, 2])$ satisfying $\chi \equiv 1$ on $[0, 1]$. Set

$$\tilde{z}_{k,\beta} = \sum_{j=0}^{\infty} \chi(\lambda_j h) z_{k,\beta}^{(j)},$$

where $\lambda_j \to \infty$, $\lambda_j < \lambda_{j+1}$ has yet to be selected. Observe for each $h > 0$, this is a finite sum, hence converges. We calculate:

$$\begin{aligned} \tilde{z}_{k,\beta} - \sum_{j=0}^{mN+m} z_{k,\beta}^{(j)} &= \sum_{mN+m+1}^{\infty} \chi(\lambda_j h) z_{k,\beta}^{(j)} + \sum_{0}^{mN+m} z_{k,\beta}^{(j)}(\chi(\lambda_j h) - 1) \\ &=: A + B. \end{aligned}$$

But since $x\chi(x)$ is uniformly bounded, we have

$$\begin{aligned} |A| &\leq \sum_{mN+m+1}^{\infty} C_j h^{(j+1)/m} \frac{\lambda_j h}{\lambda_j h} \chi(\lambda_j h) \\ &\leq \sum_{mN+m+1}^{\infty} C_j' h^{(j-m+1)/m} \lambda_j^{-1} \\ &\leq h^N \sum_{mN+1}^{\infty} 2^{-j} \end{aligned}$$

if λ_j is sufficiently large.

To estimate B, we observe for $0 < \lambda_{mN+m} h \leq 1$, $B = 0$ since $\chi \equiv 1$ on $[0, 1]$. If $\lambda_{mN+m} < h$, we calculate

$$\begin{aligned} |B| &\leq \sum_{0}^{mN+m} C_j h^{(j+1)/m} (\chi(\lambda_j h) - 1) \\ &\leq C_N h^{1/m} \lambda_{mN+m}^N h^N, \end{aligned}$$

which is (9.19).

Now for fixed (β, k) and $N > 0$, we have the crude estimate

$$\tilde{z}_{\beta,k}^l - \left(\sum_{j=0}^{mN+m} (z_{k,\beta}^{(j)}) \right)^l = \left(\tilde{z}_{\beta,k} - \sum_{j=0}^{mN+m} (z_{k,\beta}^{(j)}) \right) (l\mathcal{O}(1)),$$

which from the definitions of $z_{k,\beta}$, $\tilde{z}_{k,\beta}$, and $Q_l(h, I)$ gives:

$$\begin{aligned}
&hD_t\widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta + (Q(\tilde{z}_{k,\beta}, t) - \tilde{z}_{k,\beta})\widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta = \\
&\quad = \quad hD_t\widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta \\
&\qquad + \left(\sum_{l=0}^{mN+m} \left(\sum_{j=0}^{mN} (z_{k,\beta}^{(j)}) \right)^l Q_l(h, h(2\beta + \mathbb{1})) - \sum_{j=0}^{mN+m} (z_{k,\beta}^{(j)}) \right) \widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta \\
&\qquad + \mathcal{O}(h^N)\|v_\beta\|_{L^2(\mathbb{R}^{n-1})} \\
&\quad = \quad (2k\pi h - \tilde{z}_{k,\beta})\widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta + \mathcal{O}(h^N)\|v_\beta\|_{L^2(\mathbb{R}^{n-1})}.
\end{aligned}$$

Hence

$$\widetilde{M}(\tilde{z}_{k,\beta}, t)v_\beta = e^{it(2\pi k - \tilde{z}_{k,\beta}/h)}v_\beta + t\mathcal{O}(h^{N-1})\|v_\beta\|_{L^2(\mathbb{R}^{n-1})},$$

so

$$(e^{i\tilde{z}_{k,\beta}/h} - \widetilde{M}(\tilde{z}_{k,\beta}))v_\beta = \mathcal{O}(h^{N-1})\|v_\beta\|_{L^2(\mathbb{R}^{n-1})}$$

for any N, or

$$(e^{i\tilde{z}_{k,\beta}/h} - \widetilde{M}(\tilde{z}_{k,\beta}))v_\beta = \mathcal{O}(h^\infty)\|v_\beta\|_{L^2(\mathbb{R}^{n-1})}.$$

Now the definition of $\widetilde{M}$ implies

$$M(\tilde{z}_{k,\beta})V(\tilde{z}_{k,\beta})^{-1}v_\beta = V(\tilde{z}_{k,\beta})^{-1}v_\beta + \mathcal{O}(h^\infty)\|V(\tilde{z}_{k,\beta})^{-1}v_\beta\|_{L^2(\mathbb{R}^{n-1})},$$

so

$$u_{\tilde{z}_{k,\beta}} := E_+V(\tilde{z}_{k,\beta})^{-1}v_\beta,$$

with E_+ defined in (4.4) satisfies (9.3).

Finally, the quantization conditions (9.8-9.9) and the estimates (9.13) and (9.18) give

$$\#\{z : (Q(z, h) - z)v = \mathcal{O}(h^\infty)\} \geq C^{-1}h^{-n(1-1/m)},$$

which is (9.4).

Appendix A. Semi-hyperbolic geodesics in 3 dimensions

In this appendix, we modify the example of Colin de Verdière-Parisse [CVP] to extend to three dimensions and have a semi-hyperbolic geodesic.

Consider the Riemannian manifold

$$M = \mathbb{R}_x/\mathbb{Z} \times \mathbb{R}_y \times \mathbb{R}_z$$

equipped with the metric

$$ds^2 = \cosh^2 y(2z^4 - z^2 + 1)^2 dz^2 + dy^2 + dz^2.$$

Thus the matrix for the metric

$$g_{ij} = \begin{cases} \cosh^2 y(2z^4 - z^2 + 1)^2, & i = j = 1, \\ 1, & i = j = 2, 3, \\ 0, & i \neq j. \end{cases},$$

and we calculate the Christoffel symbols:

$$\begin{aligned} \Gamma^1_{2,1} &= \Gamma^1_{1,2} = \tanh y, \\ \Gamma^1_{3,1} &= \Gamma^1_{1,3} = (8z^3 - 2z)(2z^4 - z^2 + 1)^{-1}, \\ \Gamma^2_{1,1} &= -\sinh y \cosh y(2z^4 - z^2 + 1)^2, \\ \Gamma^3_{1,1} &= -(8z^3 - 2z)(2z^4 - z^2 + 1)\cosh^2 y, \end{aligned}$$

with all other Christoffel symbols equal to zero. The geodesic equtions are

$$\begin{aligned} \ddot{x} &= -2(\tanh y)\dot{y}\dot{x} - 2((8z^3 - 2z)(2z^4 - z^2 + 1)^{-1})\dot{z}\dot{x} \\ \ddot{y} &= \sinh y \cosh y(2z^4 - z^2 + 1)^2 \dot{x}^2 \\ \ddot{z} &= (8z^3 - 2z)(2z^4 - z^2 + 1)\cosh^2 y \dot{x}^2. \end{aligned}$$

Setting $v_x = \dot{x}$, $v_y = \dot{y}$, and $v_z = \dot{z}$, we get the first order system

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{v}_x &= -2(\tanh y)v_y v_x - 2((8z^3 - 2z)(2z^4 - z^2 + 1)^{-1})v_z v_x, \\ \dot{y} &= v_y, \\ \dot{v}_y &= \sinh y \cosh y(2z^4 - z^2 + 1)^2 v_x^2, \\ \dot{z} &= v_z, \\ \dot{v}_z &= (8z^3 - 2z)(2z^4 - z^2 + 1)\cosh^2 y v_x^2. \end{aligned}$$

There are trivially three periodic geodesics, given by the solutions

$$\begin{aligned} x(t) &= v_x(0)t + x(0), \\ y(t) &= 0, \\ z(t) &= 0, \pm 1/2. \end{aligned}$$

Next we examine the Laplace-Beltrami operator on M. We compute

$$\begin{aligned} \Delta &= |g|^{-1/2}\partial_i |g|^{1/2} g^{ij}\partial_j \\ &= \cosh^{-2} y(2z^4 - z^2 + 1)^{-2}\partial_x^2 + \partial_y^2 + \partial_z^2 \\ &\quad + \tanh y\partial_y + (8z^3 - 2z)(z^4 - z^2 + 1)^{-1}\partial_z. \end{aligned}$$

The isometry $T : L^2(M, d\mathrm{Vol}_g) \to L^2(M, dx\,dy\,dz)$ given by

$$Tu(x, y, z) = \cosh^{1/2} y(2z^4 - z^2 + 1)^{1/2} u(x, y, z)$$

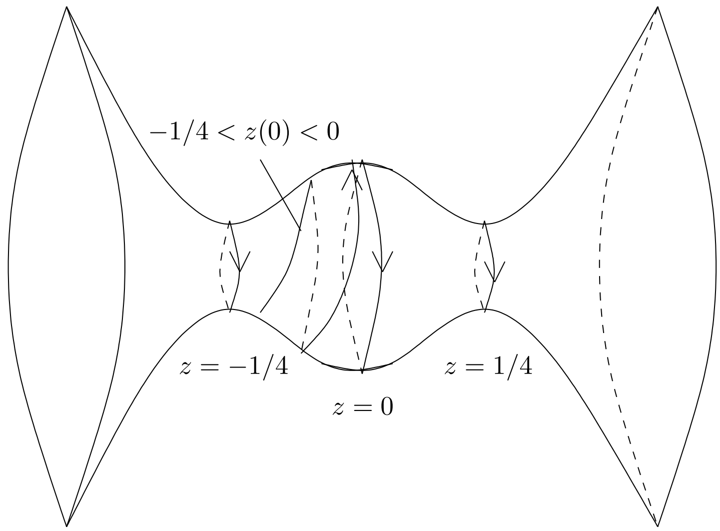

FIGURE 8. The x-z hypersurface in M with some representative orbits.

conjugates Δ into a self-adjoint operator $\widetilde{\Delta}$. A computation yields

$$\begin{aligned}\widetilde{\Delta} &= T\Delta T^{-1}\\ &= \cosh^{-2} y(2z^4 - z^2 + 1)^{-2}\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{4}(1 + \operatorname{sech}^2 y)\\ &\quad + \frac{1}{4}(8z^3 - 2z)^2(2z^4 - z^2 + 1)^{-2} - \frac{1}{2}(24z^2 - 2)(2z^4 - z^2 + 1)^{-1}.\end{aligned}$$

In keeping with the theme of this work, we want to examine asymptotic behaviour of eigenfunctions for this operator. In order to separate variables, let

$$\varphi_{k,\lambda}(x, y, z) = e^{ikx}\psi_{k,\lambda}(y, z),$$

and compute:

$$\begin{aligned}-\widetilde{\Delta}\varphi_{k,\lambda} &=\\ &= \Big(-\Delta_{y,z} + k^2 \cosh^{-2} y(2z^4 - z^2 + 1)^{-2} + \frac{1}{4}(1 + \operatorname{sech}^2 y)\\ &\quad -\frac{1}{4}(8z^3 - 2z)^2(2z^4 - z^2 + 1)^{-2} + \frac{1}{2}(24z^2 - 2)(2z^4 - z^2 + 1)^{-1}\Big)\varphi_{k,\lambda}.\end{aligned}$$

Rearranging, we have the following equation for $\psi_{k,\lambda}$:

$$\begin{aligned}&\Big(-\Delta_{y,z} + k^2(\cosh^{-2} y(2z^4 - z^2 + 1)^{-2} - 1)\\ &\quad + \frac{1}{4}(1 + \operatorname{sech}^2 y) - \frac{1}{4}(8z^3 - 2z)^2(2z^4 - z^2 + 1)^{-2}\\ &\quad + \frac{1}{2}(24z^2 - 2)(2z^4 - z^2 + 1)^{-1}\Big)\psi_{k,\lambda}\\ &\qquad\qquad = (\lambda - k^2)\psi_{k,\lambda}.\end{aligned}$$

We divide by k^2 and use $h = 1/k$ as the semiclassical parameter, giving

$$\begin{aligned} P(h)\psi_h &= (-h^2\Delta_{yz} + V(y,z))\psi_h \\ &= (h^2\lambda - 1)\psi_h, \end{aligned}$$

with

$$\begin{aligned} V(y,z) &= \cosh^{-2} y(2z^4 - z^2 + 1)^{-2} - 1 + \\ & \quad h^2 \frac{1}{4}(1 + \operatorname{sech}^2 y) - h^2 \frac{1}{4}(8z^3 - 2z)^2(2z^4 - z^2 + 1)^{-2} \\ & \quad + h^2 \frac{1}{2}(24z^2 - 2)(2z^4 - z^2 + 1)^{-1}. \end{aligned}$$

The semiclassical principal symbol is

$$\begin{aligned} \sigma_h(P) &= \eta^2 + \zeta^2 + \cosh^{-2} y(2z^4 - z^2 + 1)^{-2} - 1 \\ &=: \eta^2 + \zeta^2 + \widetilde{V}. \end{aligned}$$

Observe $\widetilde{V}$ has nondegenerate critical points at $y = 0$, $z = 0, \pm 1/4$. The signatures of $\partial^2 \widetilde{V}$ are $(-,+)$, $(-,-)$, and $(-,-)$, respectively. Thus the quadratic part of the normal forms for $\sigma_h(P)$ takes the form

$$\begin{aligned} & \lambda_1 y\eta + \frac{\lambda_2}{2}(z^2 + \zeta^2), \text{ near } y = 0, z = 0, \text{ and} \\ & \lambda_1 y\eta + \lambda_2 z\zeta, \text{ near } y = 0, z = \pm 1/4. \end{aligned}$$

References

[AbMa] Abraham, R. and Marsden, J. *Foundations of Mechanics*. W. A. Benjamin, Inc., New York, 1967.

[AuMa] Aurich, R. and Marklof, J. Trace Formulae for 3-dimensional Hyperbolic Lattices and Application to a Strongly Chaotic Tetrahedral Billiard. *Phys. D.* **92**, No. 1-2, 1996. p. 101-129.

[BoCh] Bony, J.-M. and Chemin, J.-Y. Espaces fonctionnels associés au calcul de Weyl-Hörmander. *Bull. Soc. math. France.* **122**, 1994, p. 77-118.

[Bur] Burq, N. Smoothing Effect for Schrödinger Boundary Value Problems. *Duke Math. Journal.* **123**, No. 2, 2004, p. 403-427.

[BuZw] Burq, N. and Zworski, M. Geometric Control in the Presence of a Black Box. *J. Amer. Math. Soc.* **17**, 2004, p. 443-471.

[CaPo] Cardoso, F. and Popov, G. Quasimodes with exponentially small errors associated with elliptic periodic rays. *Asymptot. Anal.* **30**, No. 3-4, 2002, p. 217-247.

[Chr1] Christianson, H. Semiclassical Non-concentration near Hyperbolic Orbits. *J. Funct. Anal.* **262** (2007), no. 2, 145–195.

[Chr1a] Christianson, H. Erratum: Semiclassical Non-concentration near hyperbolic orbits, *J. Funct. Anal.* **262** (2007), no. 2, 145–195. *preprint.* (2009), `http://www.math.mit.edu/~hans/papers/nc-erratum.pdf`

[Chr2] Christianson, H. Dispersive Estimates for Manifolds with one Trapped Orbit. *Commun. PDE*, **33** (2008) No. 7, p. 1147-1174.

[CdV] Colin de Verdière, Y. Quasi-modes sur les varits Riemanniennes. *Invent. Math.*, **43**, No. 1, 1977, p. 15-52.

[CVP] Colin de Verdière, Y. and Parisse, B. Equilibre Instable en Régime Semi-classique: I - Concentration Microlocale. *Commun. PDE.* **19**, 1994, p. 1535-1563; Equilibre Instable en Régime Semi-classique: II - conditions de Bohr-Sommerfeld. *Ann. Inst. Henri Poincaré (Phyique théorique).* **61**, 1994, 347-367.

[EvZw] Evans, L.C. and Zworski, M. *Lectures on Semiclassical Analysis.* `http://math.berkeley.edu/~evans/semiclassical.pdf`.

[HaKa] Hasselblatt, B. and Katok, A. *Introduction to the Modern Theory of Dynamical Systems.* University of Cambridge Press, Cambridge, 1995.

[HeSj] Helffer, B. and Sjöstrand, J. Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum. *Mém. Soc. Math. France.* **39**, 1989, p. 1-124.

[HoZe] Hofer, H. and Zehnder, E. *Symplectic Invariants and Hamiltonian Dynamics.* Birkhäuser Verlag, Basel, 1994.

[Hor] Hörmander, L. *The Analysis of Linear Partial Differential Operators III.* Springer-Verlag, Berlin, 1985.

[IaSj] Iantchenko, A. and Sjöstrand, J. Birkhoff Normal Forms for Fourier Integral Operators II. *American Journal of Mathematics.* **124**, 2002. p. 817-850.

[ISZ] Iantchenko, A., Sjöstrand, J., and Zworski, M. Birkhoff Normal Forms in Semiclassical Inverse Problems. *Math. Res. Lett.* **9**, 2002. p. 337-362.

[Ral] Ralson, J. Trapped rays in spherically symmetric media and poles of the scattering matrix. *Comm. Pure Appl. Math.* **24**, 1971, p. 571-582.

[Sjö] Sjöstrand, J. Geometric Bounds on the Density of Resonances for Semiclassical Problems. *Duke Mathematical Journal.* **60**, No. 1, 1990. p. 1-57.

[SjZw1] Sjöstrand, J. and Zworski, M. Quantum Monodromy and Semi-classical Trace Formulæ. *Journal de Mathématiques Pures et Appliques.* **81**, 2002. 1-33.

[SjZw2] Sjöstrand, J. and Zworski, M. Quantum Monodromy Revisited.
`http://www.math.berkeley.edu/ zworski`.

[SjZw3] Sjöstrand, J. and Zworski, M. Fractal Upper Bounds on the Density of Semiclassical Resonances.
`http://xxx.lanl.gov/pdf/math.SP/0506307`.

[TaZw] Tang, S.H. and Zworski, M. From Quasimodes to Resonances. *Math. Res. Lett.* **5**, 1998. 261-272.

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