

CLASSIFICATION OF ORDER PRESERVING ISOMORPHISMS BETWEEN ALGEBRAS OF SEMICLASSICAL OPERATORS

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ABSTRACT. Following the work of Duistermaat-Singer [DuSi] on isomorphisms of algebras of global pseudodifferential operators, we classify order preserving isomorphisms of algebras of microlocally defined semiclassical pseudodifferential operators. Specifically, we show that any such isomorphism is given by conjugation by $A = BF$, where B is a microlocally elliptic semiclassical pseudodifferential operator, and F is a microlocally unitary h -FIO associated to the graph of a local symplectic transformation.

1. INTRODUCTION

In the study of pseudodifferential operators on manifolds, there are two important regimes to keep in mind. The first is a global study of pseudodifferential operators defined using the local Fourier transform on the cotangent bundle. If X is a compact smooth manifold and T^*X is the cotangent bundle with the local coordinates $\rho = (x, \xi)$, we study pseudodifferential operators with principal symbol homogeneous at infinity in the ξ variables. Let Y be another compact smooth manifold of the same dimension as X , and suppose there is an algebra isomorphism from the algebra of all pseudodifferential operators on X (filtered by order) to the same algebra on Y , and suppose that isomorphism preserves the order of the operator. Then Duistermaat-Singer [DuSi] have shown that this isomorphism is necessarily given by conjugation by an elliptic Fourier Integral Operator (FIO).

The other setting is the semiclassical or “small- h ” regime. One can study globally defined semiclassical pseudodifferential operators, but many times it is meaningful to study operators which are microlocally defined in some small set (see §2 for definitions). Then we think of the h parameter as being comparable to $|\xi|^{-1}$ in the global, non-semiclassical regime. Thus the study of small h asymptotics in the microlocally defined regime should correspond to the study of high frequency asymptotics in the global regime. We therefore expect a similar result to that presented in [DuSi], although the techniques used in the proof will vary slightly.

Let X be a smooth manifold, $\dim X = n \geq 2$, and assume $U \subset T^*X$ is a neighbourhood of $\rho_0 \in T^*X$ so that $H^1(U, \mathbb{C}) = 0$. Let Y be another smooth manifold, $\dim Y = n$. Let $V \subset T^*Y$ be a non-empty pre-compact open subset of the cotangent bundle of Y . Let $\Psi^0/\Psi^{-\infty}(U)$ denote the algebra of semiclassical pseudodifferential operators defined microlocally in U filtered by the order in h , and similarly for V (see §2 for definitions).

Theorem 1. *Suppose*

$$g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$$

is an order preserving algebra isomorphism. For every $\tilde{U} \Subset U$ open, there is a symplectomorphism

$$\kappa : \tilde{U} \rightarrow \overline{\kappa(\tilde{U})}$$

and $h_0 > 0$ such that, if F is a microlocally unitary h -FIO associated to κ , for all $0 < h < h_0$ and all $P \in \Psi^0/\Psi^{-\infty}(U)$ we have

$$(1.1) \quad g(P) = BFPF^{-1}B^{-1} \text{ microlocally in } \kappa(\tilde{U}) \times \tilde{U},$$

where $B \in \Psi^0(V)$ is elliptic.

Remark. A few remarks are needed about Theorem 1. First, this paper concerns operators which are defined *microlocally* in a precompact open subset of T^*X (see §2 for definitions). In keeping with the theme of working up to microlocal equivalence, Theorem 1 is really the analogue of Lemma 2 in [DuSi].¹ In proving Theorem 1 of [DuSi], the authors take pains to remove the residual class $\Psi^{-\infty}$ from the statement of the Theorem, which we do not do in this note. In any applications of Theorem 1 in the present note, we would first cut off to a precompact neighbourhood so Theorem 1 applies, then fix B and F (up to microlocal equivalence) and then deal with the errors from both the residual class and the cutoffs separately.

Second, there are two places in which the assumption $H^1(U, \mathbb{C}) = 0$ is used. One is that the h -FIO F in Theorem 1 is constructed via deformation in the proof of Proposition 2.1. This deformation proof requires $H^1(U, \mathbb{C}) = 0$, although there are examples in which $H^1(U, \mathbb{C}) \neq 0$ and symplectomorphisms which *can* be quantized as an h -FIO (see Section 4 for an example). The second place this assumption is used is in the construction of the pseudodifferential operator B , where we need to conclude that a closed 1-form is exact to construct the symbol of B .

Third, as the construction of the h -FIO is by deformation and only valid in a neighbourhood in phase space, we say F is only *microlocally* defined. The operator F is unique up to $\mathcal{O}(h^2)$, and, once F is fixed, B is unique up to $\mathcal{O}(h^\infty)$.

Fourth, to put Theorem 1 in context, we observe that every algebra homomorphism of the form (1.1) is an order preserving algebra isomorphism (see Proposition 2.1 below).

Finally, automorphisms of algebras of pseudodifferential operators have been studied in the context of the more abstract Berezin-Toeplitz quantization in [Zel], as well as for the Weyl algebra on Poisson manifolds in [B-KKo].

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¹The statement of Theorem 1 is nevertheless simpler than [DuSi, Lemma 2] as in this case there are no transmissions. See the remark after Lemma 3.3 for an explanation of this.

2. PRELIMINARIES

Let $\mathcal{C}^\infty(T^*X)$ denote the algebra of smooth, \mathbb{C} -valued functions on T^*X , and define the global symbol classes

$$\mathcal{S}^m(T^*X) = \{a \in \mathcal{C}^\infty((0, h_0]_h; \mathcal{C}^\infty(T^*X)) : |\partial_{x,\xi}^\alpha a| \leq C_\alpha h^{-m}\}.$$

We define the essential support of a symbol by complement:

$$\begin{aligned} \text{ess-supp}_h(a) &= \\ &= \mathcal{C} \{(x, \xi) \in T^*X : |\partial_{x,\xi}^\alpha a| \leq C_\alpha h^N \ \forall N \text{ and } \forall (x', \xi') \text{ near } (x, \xi)\}. \end{aligned}$$

By multiplying elements of $\mathcal{S}^m(T^*X)$ by an appropriate cutoff in $\mathcal{C}_c^\infty(U)$, we may think of symbols as being microlocally defined in U , and define the class of symbols with essential support in U

$$\mathcal{S}^m(U) = \{a \in \mathcal{C}^\infty((0, 1]_h; \mathcal{C}_c^\infty(U)) : |\partial_{x,\xi}^\alpha a| \leq C_\alpha h^{-m}\}.$$

We write $\mathcal{S}^m = \mathcal{S}^m(U)$ when there is no ambiguity.

In keeping with the theme of defining everything up to microlocal equivalence, we recall Borel's lemma, which says for every sequence $a_j \in \mathcal{S}^{m_j}$, $j \geq 0$, with $m_j > m_{j+1}$ and $m_j \rightarrow -\infty$, there is an $a \in \mathcal{S}^{m_0}$ which is *asymptotic* to the sum

$$a(x, \xi; h) \sim \sum_{j=0}^{\infty} a_j(x, \xi; h).$$

Here, "asymptotic to" means that for every $N \geq 0$,

$$a(x, \xi; h) - \sum_{j=0}^N a_j(x, \xi; h) \in \mathcal{S}^{m_{N+1}}.$$

By \mathcal{S}^0/s^{-1} we denote the set of equivalence classes of symbols, where a is equivalent to b if $a - b \in \mathcal{S}^{-1}$.

We have the corresponding spaces of pseudodifferential operators $\Psi^m(U)$ acting by the local formula (Weyl calculus)

$$\text{Op}_h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a\left(\frac{x+y}{2}, \xi; h\right) e^{i\langle x-y, \xi \rangle/h} u(y) dy d\xi.$$

For $A = \text{Op}_h^w(a)$ and $B = \text{Op}_h^w(b)$, $a \in \mathcal{S}^m$, $b \in \mathcal{S}^{m'}$ we have the composition formula (see, for example, the review in [DiSj])

$$(2.1) \quad A \circ B = \text{Op}_h^w(a \# b),$$

where

$$(2.2) \quad \mathcal{S}^{m+m'} \ni a \# b(x, \xi) := e^{\frac{i\hbar}{2} \omega(D_x, D_\xi; D_y, D_\eta)} (a(x, \xi) b(y, \eta)) \Big|_{\substack{x=y \\ \xi=\eta}},$$

with ω the standard symplectic form. Observe $\#$ preserves essential support in the sense that if $\text{ess-supp}_h(a) \cap \text{ess-supp}_h(b) = \emptyset$, then $a \# b = \mathcal{O}(h^\infty)$. We define the wavefront set of a pseudodifferential operator $A = \text{Op}_h^w(a)$ as

$$\text{WF}_h(A) = \text{ess-supp}_h(a),$$

so that $\Psi^m(U)$ is the class of pseudodifferential operators with wavefront set contained in U . We denote

$$\begin{aligned}\Psi^0(U) &:= \bigcup_{m \leq 0} \Psi^m(U) \text{ and} \\ \Psi^{-\infty}(U) &:= \bigcap_{m \in \mathbb{Z}} \Psi^m(U).\end{aligned}$$

We will need the definition of microlocal equivalence of operators. Suppose $T : \mathcal{C}^\infty(X) \rightarrow \mathcal{C}^\infty(X)$ and that for any seminorm $\|\cdot\|_1$ on $\mathcal{C}^\infty(X)$ there is a second seminorm $\|\cdot\|_2$ on $\mathcal{C}^\infty(X)$ such that

$$\|Tu\|_1 = \mathcal{O}(h^{-M_0})\|u\|_2$$

for some M_0 fixed. Then we say T is *semiclassically tempered*. We assume for the rest of this paper that all operators satisfy this condition (see [EvZw, Chap. 10] for more on this). Let $U, V \subset T^*X$ be open pre-compact sets. We think of operators defined microlocally near $V \times U$ as equivalence classes of tempered operators. The equivalence relation is

$$T \sim T' \iff A(T - T')B = \mathcal{O}(h^\infty) : \mathcal{D}'(X) \rightarrow \mathcal{C}^\infty(X)$$

for any $A, B \in \Psi_h^{0,0}(X)$ such that

$$\begin{aligned}\text{WF}_h(A) \subset \tilde{V}, \quad \text{WF}_h(B) \subset \tilde{U}, \text{ with } \tilde{V}, \tilde{U} \text{ open and} \\ \bar{V} \Subset \tilde{V} \Subset T^*X, \quad \bar{U} \Subset \tilde{U} \Subset T^*X.\end{aligned}$$

In the course of this paper, when we say $P = Q$ *microlocally* near $U \times V$, we mean for any A, B as above,

$$APB - AQB = \mathcal{O}_{L^2 \rightarrow L^2}(h^\infty),$$

or in any other norm by the assumed pre-compactness of U and V . Similarly, we say $B = T^{-1}$ on $V \times V$ if $BT = I$ microlocally near $U \times U$ and $TB = I$ microlocally near $V \times U$. Thus

$$\Psi^0 / \Psi^{-\infty}(U)$$

is the algebra of bounded semiclassical pseudodifferential operators defined microlocally in U modulo this equivalence relation. It is interesting to observe that this equivalence relation has a different meaning in the high-frequency regime. There, $\Psi^{-\infty}(X)$ corresponds to smoothing operators, although they may not be “small” in the sense of $h \rightarrow 0$.

We have the principal symbol map

$$\sigma_h : \Psi^m(U) \rightarrow \mathcal{S}^m / \mathcal{S}^{m-1}(U),$$

which gives the left inverse of Op_h^w in the sense that

$$\sigma_h \circ \text{Op}_h^w : \mathcal{S}^m(U) \rightarrow \mathcal{S}^m / \mathcal{S}^{m-1}(U)$$

is the natural projection.

We will use the following well-known semiclassical version of Egorov’s theorem (see [Ch1, Ch2] or [EvZw] for a proof).

Proposition 2.1. *Suppose U is an open neighbourhood of $(0,0)$ and $\kappa : \overline{U} \rightarrow \overline{U}$ is a symplectomorphism fixing $(0,0)$. Then there is a unitary operator $F : L^2 \rightarrow L^2$ such that for all $A = \text{Op}_h^w(a)$,*

$$AF = FB \text{ microlocally on } U \times U,$$

where $B = \text{Op}_h^w(b)$ for a Weyl symbol b satisfying

$$b = \kappa^*a + \mathcal{O}(h^2).$$

F is microlocally invertible in $U \times U$ and $F^{-1}AF = B$ microlocally in $U \times U$.

Observe that Proposition 2.1 implicitly identifies a neighbourhood U with its coordinate representation. In this note, we won't use that the error is $\mathcal{O}(h^2)$; only that it is $\mathcal{O}(h)$.

Let Y be another smooth manifold of the same dimension as X , and let $V \subset T^*Y$ be a non-empty, pre-compact, open set. We say

$$g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$$

is an order preserving algebra isomorphism (of algebras filtered by powers of h) if

$$g(\Psi^m(U)) = \Psi^m(V), \quad g^{-1}(\Psi^m(V)) = \Psi^m(U),$$

and for every $A, A' \in \Psi^m(U)$, $B \in \Psi^{m'}(U)$,

$$\begin{aligned} g(A + A') &= g(A) + g(A') \text{ mod } \Psi^{-\infty}(V), \\ g(AB) &= g(A)g(B) \text{ mod } \Psi^{-\infty}(V). \end{aligned}$$

3. THE PROOF OF THEOREM 1

We break the proof of Theorem 1 into several lemmas. The first lemma classifies maximal ideals of $\mathcal{S}^0/\mathcal{S}^{-1} + \mathbb{C}$ (which is the commutative algebra $\mathcal{S}^0/\mathcal{S}^{-1}$ with a unit added) in terms of certain equivalence classes of functions.

Lemma 3.1. *The maximal ideals of $\mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C}$ are either of the form*

$$(3.1) \quad \mathcal{M}_\rho := \{p \in \mathcal{S}^0/\mathcal{S}^{-1}(U) : p(\rho) = \mathcal{O}(h), \rho \in U\},$$

or

$$(3.2) \quad \mathcal{M}_{\partial U} := \mathcal{S}^0/\mathcal{S}^{-1}(U).$$

Proof. Clearly for each $\rho \in U$, \mathcal{M}_ρ is a maximal ideal, as is $\mathcal{M}_{\partial U}$. In order to prove these are the only maximal ideals, suppose for contradiction that \mathcal{M} is another maximal ideal which is not of the form (3.1) for any $\rho \in U$. We claim then that $\mathcal{M}_{\partial U} \subset \mathcal{M}$. To prove this, we observe that for each point $\rho \in U$, there is $a_\rho \in \mathcal{M}$ such that $|a_\rho(\rho)| \geq C^{-1}$ for some $C > 0$ independent of h . Further, by multiplying by a constant if necessary, we may assume for each ρ there is a neighbourhood U_ρ of ρ such that $\text{Re } a_\rho|_{\overline{U}_\rho} \geq 1$. Let $a(x, \xi) \in \mathcal{M}_{\partial U}$, and let

$$K = \text{ess-supp}_h(a) \Subset U.$$

As K is compact, we can cover it with finitely many of the U_ρ ,

$$K \subset U_{\rho_1} \cup \dots \cup U_{\rho_m},$$

and

$$b := \sum_{j=1}^m a_{\rho_j} \in \mathcal{M}$$

satisfies $|b| \geq |\operatorname{Re} b| \geq 1$ on K . Thus $a/b \in \mathcal{M}_{\partial U}$ implies

$$a = \left(\frac{a}{b}\right) b \in \mathcal{M}.$$

Thus $\mathcal{M}_{\partial U} \subset \mathcal{M}$. But $\mathcal{M}_{\partial U}$ is maximal, so either $\mathcal{M} = \mathcal{M}_{\partial U}$ or $\mathcal{M} = \mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C}$. \square

The following three lemmas are a semiclassical version of [DuSi] with a few modifications to the proofs.

Lemma 3.2. *Suppose $g : \Psi^0/\Psi^{-\infty}(U) \rightarrow \Psi^0/\Psi^{-\infty}(V)$ is an order preserving algebra isomorphism. Then there exists a diffeomorphism $\kappa : U \rightarrow V$.*

Proof. We first “unitalize” our algebra of pseudodifferential operators by adding constant multiples of identity. That is, let

$$\tilde{\mathcal{S}}^m(U) = \{a \in \mathcal{C}^\infty((0, 1]_h; \mathcal{C}_c^\infty(U) + \mathbb{C}) : |\partial^\alpha a| \leq C_\alpha h^{-m}\},$$

and let $\tilde{\Psi}^m(U) = \operatorname{OP}\tilde{\mathcal{S}}^m(U)$. We extend g to an isomorphism

$$\tilde{g} : \tilde{\Psi}^0/\Psi^{-\infty}(U) \rightarrow \tilde{\Psi}^0/\Psi^{-\infty}(V)$$

by defining for $C \in \mathbb{C}$ and $P \in \Psi^0(U)$

$$\tilde{g}(C + P) := C + g(P).$$

Observe \tilde{g} induces an algebra isomorphism

$$g_0 : \tilde{\mathcal{S}}^0/\mathcal{S}^{-1}(U) = \mathcal{S}^0/\mathcal{S}^{-1}(U) + \mathbb{C} \rightarrow \tilde{\mathcal{S}}^0/\mathcal{S}^{-1}(V) = \mathcal{S}^0/\mathcal{S}^{-1}(V).$$

Since g_0 is an algebra isomorphism, it takes maximal ideals to maximal ideals. We observe that since $g_0 : \mathcal{S}^0/\mathcal{S}^{-1}(U) \rightarrow \mathcal{S}^0/\mathcal{S}^{-1}(V)$,

$$g_0(\mathcal{M}_{\partial U}) = \mathcal{M}_{\partial V}.$$

We can now define a map

$$\kappa : U \rightarrow V$$

by the following: for $\rho \in U$, define $\kappa(\rho)$ by

$$g_0(\mathcal{M}_\rho) = \mathcal{M}_{\kappa(\rho)}.$$

By applying g_0^{-1} , we immediately see κ is bijective.

Now for $p \in \mathcal{S}^0(U)$ and $\rho \in U$, observe

$$p - p(\rho) \cdot 1 \in \mathcal{M}_\rho$$

implies

$$g_0(p) - p(\rho) \cdot 1 \in \mathcal{M}_{\kappa(\rho)}.$$

Thus

$$g_0(p) (\kappa(\rho)) = p(\rho) + \mathcal{O}(h)$$

for every $\rho \in U$ implies

$$g_0(p) = p \circ \kappa^{-1} + \mathcal{O}_p(h),$$

where $\mathcal{O}_p(h)$ depends on p , but is smooth and uniformly bounded on U since p has compact support. Hence as a map $\mathcal{S}^0/\mathcal{S}^{-1}(U) \rightarrow \mathcal{S}^0/\mathcal{S}^{-1}(V)$,

$$g_0(p) = p \circ \kappa^{-1}.$$

For each $\rho \in U$, let (x, ξ) be local coordinates for U in a neighbourhood of ρ which does not meet ∂U . Choosing a suitable cutoff χ_ρ equal to 1 near ρ , the $\chi_\rho x_j$ and $\chi_\rho \xi_k$ are *approximate coordinates* near ρ :

$$\begin{aligned} \chi_\rho x_j, \chi_\rho \xi_k &\in \mathcal{S}^0(U) \text{ for all } j, k; \\ \chi_\rho x_j &= x_j, \chi_\rho \xi_k = \xi_k \text{ near } \rho. \end{aligned}$$

Thus

$$(\chi_\rho x_j) \circ \kappa^{-1} \in \mathcal{S}^0(V),$$

and similarly for $\chi_\rho \xi_j$ for all j . Composing with inverse coordinate functions in a neighbourhood of $\kappa(\rho)$ implies κ^{-1} is smooth on U . The same argument applied to g^{-1} shows κ is smooth on V , hence a diffeomorphism. \square

Lemma 3.3. *The diffeomorphism κ constructed in Lemma 3.2 is symplectic.*

Remark. We observe that in this Lemma, the phenomena of transmissions do not occur as in [DuSi, Lemma 2]. In that case, the diffeomorphism constructed using the point ideals as in Lemma 3.2 is a diffeomorphism $S^*X \rightarrow S^*Y$ where X and Y are n -dimensional manifolds, and the point ideals are contained in the algebra of functions which are homogeneous of degree 0. It is in proving that the diffeomorphism lifts to $T^*X \setminus \{0\}$ by studying the action on functions homogeneous of degree 1 that an ambiguity is introduced. Since we are a priori working with open sets in T^*X and smooth functions with no homogeneity assumption, no such ambiguity arises here. To see the relationship between the present result and that in [DuSi, Lemma 2], by considering symbols homogeneous of degree 1 in, say, a conic neighbourhood of a point in S^*X , and rescaling $h = |\xi|^{-1}$ we can apply the present result to get a localized version of [DuSi, Lemma 2]. But the ambiguity of the sign of ξ in the rescaling means if we want to lift κ to a diffeomorphism which commutes with multiplication in the fibers, κ becomes either a homogeneous symplectomorphism or a homogeneous symplectomorphism composed with the map $(x, \xi) \mapsto (x, -\xi)$, as in [DuSi].

Proof of Lemma 3.3. Observe $\Psi^0/\Psi^{-\infty}(U)$ is a Lie algebra with brackets $ih^{-1}[\cdot, \cdot]$, and g induces a Lie algebra isomorphism with $\Psi^0/\Psi^{-\infty}(V)$. $\mathcal{S}^0(U)$ is a Lie algebra with brackets $\{\cdot, \cdot\}$, hence g_0 is a Lie algebra isomorphism $\mathcal{S}^0/\mathcal{S}^{-1}(U) \rightarrow \mathcal{S}^0/\mathcal{S}^{-1}(V)$. Let $a, b \in \mathcal{S}^0(U)$ and calculate

$$g_0(\{a, b\}) = \{g_0(a), g_0(b)\},$$

or

$$(\{a, b\}) \circ \kappa^{-1} = \{a \circ \kappa^{-1}, b \circ \kappa^{-1}\}.$$

Letting a and b run through local approximate coordinates implies κ^{-1} is symplectic. \square

Now fix $\tilde{U} \Subset U$, and let

$$F : L^2(\tilde{U}) \rightarrow L^2(\tilde{V}) = L^2(\kappa(\tilde{U}))$$

be a microlocally unitary h -Fourier integral operator associated to κ as in Proposition 2.1. We define an automorphism of $\Psi^0/\Psi^{-\infty}(\tilde{U})$, g_1 , by

$$(3.3) \quad g_1(P) = F^{-1}g(P)F.$$

Observe g_1 is both order-preserving and preserves principal symbol.

Lemma 3.4. *Suppose*

$$g_1 : \Psi^0/\Psi^{-\infty}(\tilde{U}) \rightarrow \Psi^0/\Psi^{-\infty}(\tilde{U})$$

is an order-preserving automorphism which preserves principal symbol. Then there exists $B \in \Psi^0(\tilde{U})$, elliptic on \tilde{U} such that

$$(3.4) \quad g_1(P) = BPB^{-1} \text{ mod } \mathcal{O}(h^\infty)$$

for every $P \in \Psi^0/\Psi^{-\infty}(U)$.

Proof. The proof will be by induction, constructing B in orders of h . We then use Borel's lemma to conclude that B is asymptotic to the series in these orders of h , up to microlocal equivalence. We drop the dependence on \tilde{U} since the lemma is concerned with automorphisms. Suppose for $l \geq 1$ we have for every m and every $P \in \Psi^m$

$$g_1(P) - P \in \Psi^{m-l}.$$

This induces a map

$$\beta : \mathcal{S}^m/\mathcal{S}^{m-1} \rightarrow \mathcal{S}^{m-l}/\mathcal{S}^{m-l-1},$$

which, using the Weyl composition formula (2.1), satisfies

- (i) $\beta(pq) = \beta(p)q + p\beta(q)$;
- (ii) $\beta(\{p, q\}) = \{\beta(p), q\} + \{p, \beta(q)\}$.

Consider the action of β on \mathcal{S}^0 , and observe from property (i) above, for $p, q \in \mathcal{S}^0$,

$$\beta(pq) = \beta(p)q + p\beta(q) \in \mathcal{S}^{-l},$$

so β is h^l times a derivation on \mathcal{S}^0 .

For any $\rho \in U$, we choose coordinates (x, ξ) near ρ , and a cutoff χ_ρ which is equal to 1 near ρ and compactly supported in U . Then $\chi_\rho x_j$ and $\chi_\rho \xi_j$ become approximate coordinates which are equal to x_j and ξ_j near ρ but are in \mathcal{S}^0 . Near ρ , β takes the form

$$\beta = h^l \sum_j (\gamma_j(x, \xi) \partial_{x_j} + \delta_j(x, \xi) \partial_{\xi_j}),$$

where $\gamma_j = h^{-l} \beta(\chi_\rho x_j)$ and $\delta_j = h^{-l} \beta(\chi_\rho \xi_j)$ (here γ_j and δ_j implicitly depend on h , although we don't explicitly write it). Using property (ii) above, we have near ρ

$$\beta(\{\chi_\rho x_j, \chi_\rho \xi_k\}) = \beta(\{\chi_\rho x_j, \chi_\rho x_k\}) = \beta(\{\chi_\rho \xi_j, \chi_\rho \xi_k\}) = 0$$

which implies

$$\frac{\partial \gamma_j}{\partial x_k} = -\frac{\partial \delta_k}{\partial \xi_j}, \quad \frac{\partial \gamma_j}{\partial x_k} = \frac{\partial \gamma_k}{\partial x_j}, \quad \text{and} \quad \frac{\partial \delta_j}{\partial \xi_k} = \frac{\partial \delta_k}{\partial \xi_j}.$$

Thus there exists a locally defined smooth function f such that

$$\gamma_j = \frac{\partial f}{\partial \xi_j} \quad \text{and} \quad \delta_k = -\frac{\partial f}{\partial x_k},$$

and locally

$$h^{-l} \beta = H_f.$$

The function f can be extended to be defined on all of U by the assumption that $H^1(U, \mathbb{C}) = 0$. Define a smooth function b by

$$b = \exp(-i\chi f),$$

for a cutoff χ which is identically 1 on \tilde{U} with support in U , so that $df = idb/b$ on \tilde{U} , and

$$\beta = h^l H_{i \log b}.$$

Let $B = \text{Op}_h^w(b)$ and observe the principal symbol of

$$B^{-1}PB - P = B^{-1}[P, B]$$

in the $\mathcal{S}^0(\tilde{U})$ calculus is

$$\frac{h}{i} b^{-1} \{p, b\} = h H_{i \log b}(p).$$

For the base case of our induction, if $P \in \mathcal{S}^m(\tilde{U})$, then

$$g_1(P) - B^{-1}PB \in \mathcal{S}^{m-2}(\tilde{U}),$$

so that

$$B g_1(P) B^{-1} - P \in \mathcal{S}^{m-2}(\tilde{U}).$$

Replace $g_1(P)$ with $B g_1 B^{-1}$.

Now for the purposes of induction, assume

$$g_1(P) - P \in \mathcal{S}^{m-l}(\tilde{U}),$$

and apply the above argument to get $B_l \in \mathcal{S}^{-l}$ so that

$$g_1(P) - B_l^{-1} P B_l \in \mathcal{S}^{m-l-1}(\tilde{U}).$$

Then replacing $g_1(P)$ with $B_l g_1(P) B_l^{-1}$ finishes the induction. Thus there exists $B \in \mathcal{S}^0(\tilde{U})$ so that

$$B g_1(P) B^{-1} = P \text{ mod } \mathcal{O}(h^\infty).$$

□

Theorem 1 now follows immediately from applying Proposition 2.1 to (3.3) and (3.4).

4. AN EXAMPLE

The current paper (and indeed the paper of Duistermaat-Singer [DuSi] on which this paper is based) largely skirts a very important related question by assuming $H^1(U, \mathbb{C}) = 0$. In this section we briefly consider a simple example which helps illustrate the necessity of this condition in general. We consider the following simple question: given a symplectomorphism $\kappa : U \rightarrow \kappa(U)$, is there a unitary operator which quantizes κ ? More precisely, we investigate whether or not there is a unitary h -FIO which satisfies a Egorov type law.

Specifically, if $\kappa : U \rightarrow \kappa(U)$ is a symplectomorphism, we say a (microlocally) unitary² h -FIO F is of *Egorov type* if it satisfies a sharp Egorov conjugation law in the Weyl calculus: for $A \in \Psi_h^{l,0}$,

$$A(x, hD)F = FB(x, hD)$$

²Here “unitary” refers to unitary on a fixed Hilbert space, typically L^2 of the base space, or of a compact subset thereof. “Microlocally” refers to the possibility that the Hilbert space is weighted with some pseudodifferential cutoffs.

where $\sigma(B) = \kappa^* \sigma(A) + e_A$, with

$$e_A \in \mathcal{S}^{l-2, -2},$$

with microsupport contained in the microsupport of $\kappa^* \sigma(A)$ (see [Ch1] for notation and definitions). In the microlocal setting discussed in the previous sections, U is precompact, so this just means $\mathcal{S}^{l,m}(U) = \mathcal{S}^m(U)$ in the notation from the previous sections, and hence $e_A = \mathcal{O}(h^2)$.

Proposition 4.1. *Suppose $X = \mathbb{S}^1$, $U = T^*X \simeq \mathbb{S}_x^1 \times \mathbb{R}_\xi$ equipped with the standard symplectic structure, and*

$$\kappa(x, \xi) = (x, \xi + \beta)$$

for $\beta \in \mathbb{R}$. Then there exists an h -FIO F of Egorov type quantizing κ , unitary on $L^2(\mathbb{S}^1)$, if and only if $\beta \in h\mathbb{Z}$.

Remark. Pullback by the symplectomorphism κ from Proposition 4.1 clearly defines an order-preserving automorphism of the symbol algebra, preserving both the order in h and in ξ . If we microlocalize everything in the following proof, that means it is order-preserving in h as discussed in the previous sections.

Proposition 4.1 should come as no surprise. In fact, from the point of view of geometric quantization, there is a pre-quantization for every element in $H^1(U, \mathbb{T}^1)$, the first cohomology group with coefficients in the unitary group. They differ by an overall phase factor of $e^{i2\pi\beta/h}$ for $0 \leq \beta < h$. That is, for $\beta \notin h\mathbb{Z}$, conjugation by a unitary h -FIO quantizing κ is an isomorphism between *inequivalent pre-quantizations*, and hence should not be an automorphism.

Proof. If $\beta \in h\mathbb{Z}$, then there is a global generating function for κ , and it is trivial to quantize κ . In fact, a unitary h -FIO quantizing κ is given by multiplication by $e^{i\beta x/h}$, which is certainly 2π -periodic, unitary, and satisfies an exact Egorov type transformation law (see (4.2) below).

The proof of the reverse implication proceeds by contradiction. If there is such an operator, we lift it to the universal covering space and, by an approximate uniqueness result, we conclude that the operator must be given by multiplication by $e^{i\beta(x+2\pi j)/h}$ for some $j \in \mathbb{Z}$, modulo a lower order term. But multiplication by $e^{i\beta(x+2\pi j)/h}$ preserves periodicity of a function if and only if $\beta \in h\mathbb{Z}$.

For the purposes of contradiction, assume $\beta \notin h\mathbb{Z}$ and suppose there is nevertheless a unitary Egorov-type h -FIO F quantizing κ . That is, for $u(x) \in L^2(\mathbb{S}^1)$, $Fu \in L^2(\mathbb{S}^1)$ with $\|Fu\| = \|u\|$, $F^* = F^{-1}$, and for all pseudodifferential operators $A \in \Psi_h^{l,0}(\mathbb{S}^1)$, there is a pseudodifferential operator $e_A \in \Psi_h^{l-2,-2}(\mathbb{S}^1)$ so that

$$AF = F(\tilde{A} + e_A),$$

where $\sigma(\tilde{A})(x, \xi) = \kappa^* \sigma(A)(x, \xi) = \sigma(A)(x, \xi + \beta)$. The universal covering space of \mathbb{S}^1 is \mathbb{R} , so we want to compare F to operators acting (locally) on $L^2(\mathbb{R})$. We regard $L^2(\mathbb{S}^1)$ as a vector subspace of $\mathcal{S}'(\mathbb{R})$, that is, 2π -periodic functions on the real line which are square-integrable over one period. We lift F to an operator on $L^2(\mathbb{S}^1) \subset \mathcal{S}'(\mathbb{R})$ which is unitary over any interval of length 2π by partitioning \mathbb{R} into intervals of length bounded by 2π . That is, the lift of F acting on a function supported in an interval of length 2π is F acting on the 2π periodic extension. This is well-defined up to smoothing operators of order h^∞ according to the assumption that F is of Egorov type. We abuse notation and continue to call this lift F .

Fix $s_0 \in \mathbb{R}$, and let $a(x) \in \mathcal{C}_c^\infty([s_0, s_0 + 2\pi])$. Since a is supported in one period, we have a Egorov transformation rule:

$$(4.1) \quad a(x)F = F(a(x) + e_a)$$

for $e_a \in \Psi_h^{-2,-2}(\mathbb{S}^1)$, since κ is identity in the x variable. The operator $a(x)F$ now maps $L^2(\mathbb{S}^1)$ to $L^2([s_0, s_0 + 2\pi])$ (with no periodic extension), and by (4.1), $a(x)F$ also maps $L^2([s_0, s_0 + 2\pi])$ into $L^2([s_0, s_0 + 2\pi])$.

Consider now the multiplication operators

$$F_j = e^{i\beta(x+2\pi j)/h}$$

for $j \in \mathbb{Z}$, acting on $L^2(\mathbb{R})$. Clearly F_j is unitary on $L^2(I)$ for any interval $I \subset \mathbb{R}$, and F_j satisfies the exact Egorov law

$$(4.2) \quad \begin{aligned} A(x, hD)F_j u(x) &= (2\pi h)^{-1} \int e^{i(x-y)\xi/h} A((x+y)/2, \xi) e^{i\beta(y+2\pi j)/h} u(y) dy d\xi \\ &= (2\pi h)^{-1} e^{i\beta(x+2\pi j)/h} \int e^{i(x-y)(\xi-\beta)/h} A((x+y)/2, \xi) u(y) dy d\xi \\ &= F_j A(x, hD + \beta) u(x). \end{aligned}$$

Multiplying by the compactly supported function $a(x)$, we have

$$aF_j = F_j a$$

with no error term. If we choose a real,

$$(aF_j)^* = F_j^* a = e^{-i\beta(x+2\pi j)/h} a = a e^{-i\beta(x+2\pi j)/h},$$

so that

$$\begin{aligned} (aF)(aF_j)^* &= aF a F_j^* \\ &= a^2 F F_j^* - aF e_a F_j^* \\ &= (a^2 - a\tilde{e}_a) F F_j^* \end{aligned}$$

for a pseudodifferential operator $\tilde{e}_a \in \Psi_h^{-2,-2}([s_0, s_0 + 2\pi])$. We now claim that there is a constant c_j , $|c_j| = 1$, and a pseudodifferential operator $E_{a,j} \in \Psi_h^{-2,-2}(\mathbb{R})$ so that

$$a^2 F F_j^* = c_j a^2 (I + E_{a,j}).$$

as operators on $L^2([s_0, s_0 + 2\pi])$. Indeed, we have $a^2 F F_j^* \in \Psi_h^{0,0}(\mathbb{R})$ is a semiclassical pseudodifferential operator,

$$a^2 F F_j^* x = a^2 (x + e_1) F F_j^*, \quad e_1 \in \Psi_h^{-2,-2}(\mathbb{R}),$$

and

$$a^2 F F_j^* hD = a^2 (hD + e_2) F F_j^*, \quad e_2 \in \Psi_h^{-1,-2}(\mathbb{R}),$$

which implies the principal and sub-principal symbol of $a^2 F F_j^*$ is a constant times a^2 . Since $(F F_j^*)^* = (F F_j^*)^{-1}$ on $\text{supp}(a^2)$, we know this constant has modulus 1.

This tells us that, at least for part of a period, F is multiplication by $e^{i\beta(x+2\pi j)/h}$ for some j , up to errors of size h^2 . The next step is to show that this is true for a *whole* period. That is, let $a_1 + a_2 \equiv 1$ on $[0, 2\pi]$, but with each a_l supported in

an interval of length less than 2π so that the above discussion applies to each a_l . Then we calculate

$$\begin{aligned}
 (4.3) \quad & (a_1 + a_2)F(c_1 a_1 F_{j_1} + c_2 a_2 F_{j_2})^* \\
 &= (a_1 + a_2)F(\overline{c_1} a_1 F_{j_1}^* + \overline{c_2} a_2 F_{j_2}^*) \\
 &= (a_1 + a_2)F(\overline{c_1} a_1 e^{-i2\pi\beta j_1/h} F_0^* + \overline{c_2} a_2 e^{-i2\pi\beta j_2/h} F_0^*) \\
 &= (a_1 + a_2)(\overline{c_1} a_1 e^{-i2\pi\beta j_1/h} + \overline{c_2} a_2 e^{-i2\pi\beta j_2/h}) F F_0^*
 \end{aligned}$$

plus errors of size h^2 which we neglect for the rest of the calculation. This operator now a priori is defined on $L^2(\mathbb{R})$, but if we compose with another function b supported in $[0, 2\pi]$, but $b \equiv 1$ in a neighbourhood of $\text{supp}(a_1) \cap \text{supp}(a_2)$, we conclude we must have

$$(4.4) \quad (\overline{c_1} a_1 e^{-i2\pi\beta j_1/h} + \overline{c_2} a_2 e^{-i2\pi\beta j_2/h}) = 1 + \mathcal{O}(h^2)$$

on $\{a_1 + a_2 = 1\}$. That is, replacing a_2 with $1 - a_1$ on $\text{supp}(a_1) \cap \text{supp}(a_2)$, (4.4) must be independent of a_1 up to $\mathcal{O}(h^2)$, so that

$$\frac{c_1}{c_2} = e^{i2\pi\beta(j_2 - j_1)/h} (1 + \mathcal{O}(h^2)),$$

or c_1 and c_2 differ to leading order only by a phase shift. Without loss of generality, we write $c_1 = 1 + \mathcal{O}(h^2)$ and $c_2 = e^{-i2\pi\beta j/h} (1 + \mathcal{O}(h^2))$ in the calculation leading to (4.3) to get

$$(a_1 + a_2)F(a_1 F_0 + c_2 a_2 F_0)^* = (a_1 + a_2)(a_1 + e^{i2\pi\beta j/h} a_2) F F_0^*.$$

But by the above argument this implies that either $j = 0$ or $\beta = \mathcal{O}(h^3)$.

This implies the h -FIO F is equal to a unimodular constant times one of the F_j on any interval of length 2π , which is a whole period, and hence F cannot be periodic. □

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