DISPERSIVE ESTIMATES FOR MANIFOLDS WITH ONE TRAPPED ORBIT

HANS CHRISTIANSON

Abstract. For a large class of complete, non-compact Riemannian manifolds, (M, g), with boundary, we prove high energy resolvent estimates in the case where there is one trapped hyperbolic geodesic. As an application, we have the following local smoothing estimate for the Schrödinger propagator:

$$\int_0^T \left\| \rho_s e^{it(\Delta_g - V)} u_0 \right\|^2_{H^{1/2-\epsilon}(M)} dt \leq C_T \|u_0\|^2_{L^2(M)},$$

where $\rho_s(x) \in \mathcal{C}^\infty(M)$ satisfies $\rho_s = \langle \text{dist}_g(x, x_0) \rangle^{-s}$, $s > \frac{1}{2}$, and $V \in \mathcal{C}^\infty(M)$, $0 \leq V \leq C$ satisfies $|\nabla V| \leq C \langle \text{dist}(x, x_0) \rangle^{-1-\delta}$ for some $\delta > 0$. From the local smoothing estimate, we deduce a family of Strichartz-type estimates, which are used to prove two well-posedness results for the nonlinear Schrödinger equation.

As a second application, we prove the following sub-exponential local energy decay estimate for solutions to the wave equation when $\dim M = n \geq 3$ is odd and M is equal to $\mathbb{R}^n$ outside a compact set:

$$\begin{aligned} &\int_M |\psi \partial_t u|^2 + |\psi \nabla u|^2 \, dx \\ &\quad \leq \quad C e^{-t^{1/2}/C} \left(\|u(x,0)\|^2_{H^{1+\epsilon}(M)} + \|D_t u(x,0)\|^2_{H^\epsilon(M)} \right), \end{aligned}$$

where $\psi \in \mathcal{C}^\infty(M)$, $\psi \equiv e^{-|x|^2}$ outside a compact set.

1. Introduction

In this note we show how the results of [Chr1, Chr2] on cutoff resolvent estimates near closed hyperbolic orbits can be combined with the non-trapping resolvent estimates in [CPV] to obtain resolvent bounds in the case of one trapped hyperbolic orbit with a logarithmic loss. As applications, we prove local smoothing estimates for solutions to the linear Schrödinger equation (Theorem 1) and local energy decay estimates for solutions to the linear wave equation (Theorem 2). These theorems have direct applications to the nonlinear Schrödinger and wave equations.

We prove the high-energy resolvent estimates for a much more general class of manifolds, then specialize to the case of asymptotically Euclidean manifolds for the applications. The class of manifolds we consider for the high-energy estimates are the same as those studied (in the non-trapping case) in [CPV]. More precisely, let (M, g) be a connected Riemannian manifold, $M = X_0 \cup X$, where X_0 is a compact, connected n-dimensional Riemannian manifold and $X = [r_0, +\infty) \times S$, $r_0 \gg 1$, where S is a compact, connected $(n-1)$-dimensional Riemannian manifold without boundary. We assume ∂X_0 is compact and that X and X_0 satisfy

$$\partial X_0 = \partial M \cup \partial X, \;\; \partial M \cap \partial X = \emptyset.$$

Key words and phrases. local smoothing, local energy decay, trapping geometry, semiclassical resolvent.

We assume the metric $g|_{X_0}$ is a $\mathcal{C}^\infty$ metric on $\overline{X_0}$ and

$$g|_X = dr^2 + \sigma(r),$$

where $\sigma(r)$ is a family of smooth Riemannian metrics on S depending smoothly on r. In local coordinates, the metric $\sigma(r)$ takes the form

$$\sigma(r) = \sum_{i,j=1}^{n-1} g_{ij}(r,\theta) d\theta^i d\theta^j,$$

and if we set $X_r = [r, +\infty) \times S$, we can identify $\partial X_r \simeq (S, \sigma(r))$. Thus with $b = (\det g_{ij})^{\frac{1}{2}}$ and $(g^{ij}) = (g_{ij})^{-1}$, we have

$$\Delta_{\partial X_r} = -b^{-1} \sum_{i,j} \partial_{\theta_i}(b g^{ij} \partial_{\theta_j}),$$

and

$$\Delta_X = -b^{-1}\partial_r(b\partial_r) + \Delta_{\partial X_r}.$$

As in the introduction of [CPV], a calculation shows

$$b^{\frac{1}{2}} \Delta_X b^{-\frac{1}{2}} = \partial_r^2 + \Lambda_r + q(r,\theta),$$

with

$$\Lambda_r = -\sum_{i,j} \partial_{\theta_i}(g^{ij}\partial_{\theta_j}),$$

and

$$q(r,\theta) = (2b)^{-2}\left(\frac{\partial b}{\partial r}\right)^2 + (2b)^{-2}\sum_{i,j}\frac{\partial b}{\partial \theta_i}\frac{\partial b}{\partial \theta_j} g^{ij} + \frac{1}{2} b \Delta_X (b^{-1}).$$

We assume $q(r,\theta) = q_1(r,\theta) + q_2(r,\theta)$, where

$$|q_1(r,\theta)| \leq C, \quad \left|\frac{\partial^k q_1}{\partial r^k}\right| \leq C r^{-k-\delta} \text{ for } k \geq 1, \text{ and}$$

$$\left|\frac{\partial^{k'} q_2(r,\theta)}{\partial r^{k'}}\right| \leq C r^{-k'-1-\delta} \text{ for } k' \geq 0,$$

for $C, \delta > 0$. Observe this is satisfied for Euclidean space using a polar decomposition outside of a ball of radius r_0 (where $b = r^{n-1}\alpha(\theta)$), and for asymptotically Euclidean or conic manifolds. Define $h \in \mathcal{C}^\infty([r_0, +\infty) \times T^*(\partial X_r))$ by

$$h(r,\theta,\xi) = \sum_{i,j} g^{ij}(r,\theta)\xi_i\xi_j,$$

and assume there is a constant $C > 0$ such that for all $(\theta,\xi) \in T^*(\partial X_r)$,

$$-\frac{\partial h}{\partial r}(r,\theta,\xi) \geq \frac{C}{r} h(r,\theta,\xi).$$

Let $-\Delta_g$ be the Laplace-Beltrami operator acting on functions, with Dirichlet boundary conditions if $\partial M \neq \emptyset$, and suppose $V \in \mathcal{C}^\infty(M)$, $0 \leq V \leq C$ satisfies

$$|\nabla V| \leq C \langle \mathrm{dist}\,(x, x_0)\rangle^{-1-\delta} \tag{1.1}$$

for some $\delta > 0$.

The operator $P := -\Delta_g + V(x)$ is an unbounded operator

$$P : \mathcal{H} \to \mathcal{H},$$

where $\mathcal{H} = L^2(M)$, with domain $H^2(M)$ or $H^2(M) \cap H^1_0(M)$ in the case $\partial M \neq \emptyset$. In order to study the operator

$$P - \tau$$

for $\tau \in \mathbb{C}$ in some neighbourhood of $\mathbb{R}$, we use the following semiclassical rescaling for $-\Delta_g$. For $z \in [E-\delta, E+\delta] + i(-c_0 h, c_0 h)$ write

$$\tau = \frac{z}{h^2}.$$

Then

$$\begin{aligned} -\Delta_g + V(x) - \tau &= -\Delta_g + V(x) - \frac{z}{h^2} \\ &= \frac{1}{h^2}(-h^2\Delta_g + h^2 V(x) - z). \end{aligned}$$

Now let $P(h) = -h^2\Delta_g + h^2V(x)$ be the self-adjoint semiclassical Schrödinger operator acting on $\mathcal{H}$ with Dirichlet boundary conditions if $\partial M \neq \emptyset$. Let $p = \sigma_h(P(h))$ be the semiclassical (Weyl) principal symbol of $P(h)$ (see [EvZw, Theorem D.3]). We assume the Hamiltonian flow of H_p generates a single closed hyperbolic orbit γ in the energy level $\{p = E\}$, $E > 0$. The assumption that γ be hyperbolic means the linearization of the Poincaré map has no eigenvalues on the unit circle (see [Chr1, Chr2] for definitions). Let $\pi : T^*M \to M$ denote the natural projection, and assume that the projected generalized geodesic $\pi(\gamma)$ lies entirely within $U_0 \Subset U \Subset X_0$. If $\pi(\gamma)$ intersects ∂M, assume that the intersection is transversal. Assume further that the geometry is non-trapping outside U_0. That is, for every compact subset $K \Subset M \setminus U_0$, there is a time $T(K)$ so that if $\eta(t)$ is a generalized geodesic with $\eta(0) \in K$, there is a time $0 < \tau \leq T(K)$ such that $\eta(\pm\tau) \in (M \setminus U_0) \setminus K$.

1.1. **The Main Results.** The following theorem is our local smoothing result for solutions to the linear Schrödinger equation, and is a generalization of the results in [Bur2] and the references cited therein. The Schrödinger propagator $e^{it(\Delta_g - V(x))}$ is a unitary operator on $L^2(M)$, but this theorem says if we integrate in time, we gain some regularity.

Theorem 1. *Suppose (M, g) is a Riemannian manifold (with or without boundary) which satisfies the above assumptions, $\gamma \subset M$ is a closed hyperbolic geodesic, and $-\Delta_g$ is the Laplace-Beltrami operator (with Dirichlet boundary conditions if $\partial M \neq \emptyset$). Then for each $\epsilon > 0$ and $T > 0$, there is a constant C such that*

$$\int_0^T \left\| \rho_s e^{it(\Delta_g - V(x))} u_0 \right\|^2_{H^{1/2-\epsilon}(M)} dt \leq C \|u_0\|^2_{L^2(M)}, \tag{1.2}$$

where $\rho_s \in \mathcal{C}^\infty(M)$ satisfies

$$\rho_s(x) \equiv \langle d_g(x, x_0) \rangle^{-s} \tag{1.3}$$

for x_0 fixed and x outside a compact set, and $V \in \mathcal{C}^\infty(M)$, $0 \leq V \leq C$ satisfies (1.1).

Remark 1.1. We will see that in some cases the weighted resolvent has no poles on the real axis, and we can conclude the estimate (1.2) is global in time at the expense of replacing ρ_s with super-exponentially decreasing weights. That is, in these cases we have

$$\int_0^\infty \left\| \psi e^{it(\Delta_g - V)} u_0 \right\|^2_{H^{1/2-\epsilon}(M)} dt \leq C \|u_0\|^2_{L^2(M)}, \tag{1.4}$$

where $\psi \equiv \exp(-\mathrm{dist}_g(x, x_0)^2)$ outside a compact set. This is the case, for example, if g is an asymptotically Euclidean scattering metric, $V \equiv 0$, and $\partial M = \emptyset$ (see [Mel, Theorem 3, §10]). It is also the case if (M, g) is equal to $\mathbb{R}^n$ outside a compact set, $n \geq 2$, and V satisfies (1.6) below (see [Vai, Theorem 8, Ch.9]). See Remarks 2.1 and 3.2.

As a second application, we study solutions to the linear wave equation on (M, g):

$$\begin{cases} (-D_t^2 - \Delta_g + V(x))u(x,t) = 0, \ \ (x,t) \in M \times [0, \infty) \\ u(x, 0) = u_0 \in H^1(M), \ \ D_t u(x, 0) = u_1 \in L^2(M), \end{cases} \tag{1.5}$$

where $-\Delta_g$ is the Dirichlet Laplace-Beltrami operator on functions and $V \in \mathcal{C}^\infty(M)$ satisfies

$$\exp(\mathrm{dist}_g(x, x_0)^2) V = o(1). \tag{1.6}$$

Let $\psi \in \mathcal{C}^\infty(M)$ satisfy

$$\psi \equiv \exp(-\mathrm{dist}_g(x, x_0)^2) \tag{1.7}$$

for x outside a compact set and x_0 fixed. For u satisfying (1.5), we define the *local energy*, $E_\psi(t)$, to be

$$E_\psi(t) = \frac{1}{2} \left(\|\psi \partial_t u\|^2_{L^2(M)} + \|\psi u\|^2_{H^1(M)} \right).$$

Theorem 2. *Suppose (M, g) is equal to $\mathbb{R}^n$ outside a compact set, $n = \dim M \geq 3$ is odd, and $\gamma \subset M$ is a hyperbolic trapped ray with no other trapping. Then for each $\epsilon > 0$ and each*

$$\begin{aligned} u_0 &\in \mathcal{C}_c^\infty(M) \cap H^{1+\epsilon}(M), \ \ \textit{and} \\ u_1 &\in \mathcal{C}_c^\infty(M) \cap H^{\epsilon}(M), \end{aligned}$$

there is a constant $C > 0$ such that

$$E_\psi(t) \leq C e^{-t^{1/2}/C} \left(\|u_0\|^2_{H^{1+\epsilon}(M)} + \|u_1\|^2_{H^\epsilon(M)} \right). \tag{1.8}$$

Here the constant C depends only on $\epsilon > 0$, g, n, ψ, and the support of u_0 and u_1.

Remark 1.2. The estimate (1.8) holds whenever the resolvent admits a meromorphic extension to $\mathbb{C}$ with no poles in a complex neighbourhood of an interval $[-C, C] \subset \mathbb{R}$, which holds also, for example, if (M, g) is an exterior domain in $\mathbb{R}^n$ with $n \geq 3$ odd.

The problem of "local smoothing" estimates for the Schrödinger equation has a long history. The sharpest results to date are those of Doi [Doi] and Burq [Bur2]. Doi proved if M is asymptotically Euclidean, then one has the estimate

$$\int_0^T \left\| \psi e^{it\Delta_g} u_0 \right\|^2_{H^{1/2}(M)} dt \leq C \|u_0\|^2_{L^2(M)} \tag{1.9}$$

for $\psi \in \mathcal{C}_c^\infty(M)$ if and only if there are no trapped sets. Burq's paper showed if there is trapping due to the presence of several convex obstacles in $\mathbb{R}^n$ satisfying certain assumptions, then one has the estimate (1.9) with the $H^{1/2}$ norm replaced by $H^{1/2-\epsilon}$ for $\epsilon > 0$.

The estimates with the $\epsilon > 0$ loss in trapping geometries corresponds to a logarithmic loss in resolvent estimates for these geometries (see Theorem 3). With more care, one could replace the $\epsilon > 0$ loss in derivative with a logarithmic loss in derivative, which may help in certain applications. The proof of Theorem 3 uses a semiclassical reduction to consider an operator of the form

$$P(h) - z = -h^2\Delta_g - z,$$

with $z \in [E-\delta, E+\delta] + i(-c_0h, c_0h)$ for $E, \delta > 0$. It is shown in [Chr1, Chr2] that for $A \in \Psi_h^{0,0}(M)$ with sufficiently small wavefront set near $\gamma \subset \{p = E\}$, if $|\operatorname{Im} z| \leq c_0' h/\log(1/h)$, then

$$\|(P(h) - z)Au\|_{L^2} \geq C^{-1}\frac{h}{\log(1/h)}\|Au\|_{L^2}. \tag{1.10}$$

We will use the main results from [CPV] and propagation of singularities to extend this to an estimate on M.

As an application of Theorem 1 and the non-trapping Strichartz estimates of [HTW], we study the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + (\Delta_g - V(x))u = F(u) \text{ on } I \times M; \\ u(0,x) = u_0(x), \end{cases} \tag{1.11}$$

where $I \subset \mathbb{R}$ is an interval containing 0 and $V \in \mathcal{C}_c^\infty(M)$, $V \geq 0$. Here the nonlinearity F satisfies

$$F(u) = G'(|u|^2)u,$$

and $G : \mathbb{R} \to \mathbb{R}$ is at least C^3 and satisfies

$$|G^{(k)}(t)| \leq C_k\langle t\rangle^{\beta-k},$$

for some $\beta \geq \frac{1}{2}$.

In §4 we prove a family of Strichartz-type estimates which will result in the following local well-posedness proposition. (See §4 also for comments on optimality.) For the statement of the proposition, let $H_D^1(M)$ denote the domain of $(1-\Delta_g)^{\frac{1}{2}}$ with Dirichlet boundary conditions if $\partial M \neq \emptyset$, so that $H_D^1(M) = H_0^1(M)$, and write $H_D^s(M)$ for the domain of $(1-\Delta_g)^{s/2}$ (with Dirichlet boundary conditions if $\partial M \neq \emptyset$).

Proposition 1.3. *Suppose (M, g) satisfies the above assumptions, $V \in \mathcal{C}_c^\infty(M)$, and in addition M is asymptotically conic (as defined in* [HTW]*). Then for each*

$$s > \frac{n}{2} - \frac{k}{\max\{2\beta - 2, 2\}} \tag{1.12}$$

and each $u_0 \in H_D^s(M)$ there exists $p > \max\{2\beta - 2, 2\}$ and $0 < T \leq 1$ such that (1.11) *has a unique solution*

$$u \in C([-T,T]; H_D^s(M)) \cap L^p([-T,T]; L^\infty(M)). \tag{1.13}$$

Here $k = 1$ if $\partial M \neq \emptyset$ and $k = 2$ if $\partial M = \emptyset$.

Moreover, the map $u_0(x) \mapsto u(t,x) \in C([-T,T]; H_D^s(M))$ is Lipschitz continuous on bounded sets of $H_D^s(M)$, and if $\|u_0\|_{H_D^s}$ is bounded, T is bounded from below.

If we have H^1 energy conservation, Proposition 1.3 implies u extends to a global solution.

Corollary 1.4. *Suppose (M, g) and V satisfy the assumptions of Proposition 1.3, and assume $n \leq 2$. If $G(r) \to +\infty$ as $r \to +\infty$ then u in* (1.13) *extends to a solution*

$$u \in C((-\infty, \infty); H^1_D(M)) \cap L^p((-\infty, \infty); L^\infty(M)).$$

If $\partial M = \emptyset$, $n \leq 3$, $\beta < 3$, and $G(r) \to +\infty$ as $r \to +\infty$, then the same conclusion holds.

Remark 1.5. In particular, the cubic defocusing non-linear Schrödinger equation is globally well-posed. Observe also that three spatial dimensions is the smallest dimension in which the periodic orbit γ can have a Poincaré map whose linearization possesses complex eigenvalues.

Local energy decay for solutions to the linear wave equation has also enjoyed a long history. Studied in non-trapping exterior domains by Morawetz [Mor], Morawetz-Phillips [MoPh], and Morawetz-Ralston-Strauss [MRS], and generalized by, for example Vodev [Vod], it is well-known (see [Ral]) that when there *are* trapped rays, one cannot expect exponential decay of the energy with no loss in regularity. Metcalfe-Sogge [MeSo2] have recently shown that if there are trapped hyperbolic rays *and* sub-exponential energy decay with loss in derivative, then one has long-time existence for certain classes of quasi-linear wave equations in $\mathbb{R}^n$. Theorem 2 says this always happens with one trapped hyperbolic orbit. Specifically, suppose $M = \mathbb{R}^n \setminus U$ for $U \Subset \mathbb{R}^n$, $-\Delta$ is the Dirichlet Laplacian,

$$Q(z, w) \in \mathcal{C}^\infty(\mathbb{C}^n \times \mathbb{C}^{n^2})$$

satisfies

i) Q is linear in w,

ii) For each w, $Q(\cdot, w)$ is a symmetric quadratic form,

and consider the following initial value problem:

$$\begin{cases} (-D_t^2 - \Delta)u = Q(Du, D^2u) \text{ on } M \times [0, \infty), \\ u(x, 0) = u_0, \; D_t u(x, 0) = u_1. \end{cases} \tag{1.14}$$

The following Proposition then follows directly from [MeSo2, Theorem 1.1] in dimension $n = 3$ and [MeSo1, Theorem 1.1] in dimensions $n \geq 5$.

Proposition 1.6. *Suppose $(u_0, u_1) \in (\mathcal{C}^\infty(\mathbb{R}^n \setminus U))^2$, $n \geq 3$ odd, satisfy the compatibility condition from* [MeSo2, §1]*, and $\gamma \subset (\mathbb{R}^n \setminus U)$ is a trapped hyperbolic geodesic, with no other trapping. Assume further that if $n = 3$, the null condition* [MeSo2, (1.9), (1.10)] *holds. Then there exist $\epsilon_0 > 0$ and $N > 0$ such that for every $\epsilon \leq \epsilon_0$, if*

$$\sum_{|\alpha| \leq N} \| \langle x \rangle^{|\alpha|} \partial_x^\alpha u_0 \|_{L^2} + \sum_{|\alpha| \leq N-1} \| \langle x \rangle^{|\alpha|+1} \partial_x^\alpha u_1 \|_{L^2} \leq \epsilon,$$

then (1.14) *has a unique solution $u \in \mathcal{C}^\infty([0, \infty) \times \mathbb{R}^n \setminus U)$.*

Acknowledgments. The author would like to thank Maciej Zworski for suggesting the local smoothing problem, as well as providing much help and support during the writing of this paper. He would also like to thank Jason Metcalfe for helpful comments and suggesting Proposition 1.6, Daniel Tataru for helpful discussions about optimality of Strichartz estimates, and Nicolas Burq for suggesting Proposition 4.9. The bulk of this paper was written while the author was a graduate student at UC-Berkeley, so he would like to thank the Mathematics Department at UC-Berkeley for their support. Finally, he would like to thank the anonymous referee whose many comments and suggestions helped improve the exposition.

2. Resolvent estimates

Let

$$(P-\tau)^{-1} = (-\Delta_g + V(x) - \tau)^{-1}$$

be the classical resolvent. In this note we use the notation τ for the unsquared spectral parameter and $\lambda^2 = \tau$ for the squared parameter. It will be convenient to use the lower half-plane as the physical half-plane. The proof of Theorem 1 relies on the weighted resolvent estimates of the following Theorem.

Theorem 3. *Suppose (M, g) satisfies all of the assumptions above. Then for each $\epsilon > 0$ sufficiently small and each $s > \frac{1}{2}$ there is a constant $C > 0$ such that*

$$\left\| \rho_s (P - (\tau \pm i\epsilon))^{-1} \rho_s \right\|_{\mathcal{H} \to \mathcal{H}} \leq C \frac{\log(2 + |\tau|)}{\langle \tau \rangle^{1/2}}, \quad \tau \in \mathbb{R}. \tag{2.1}$$

Remark 2.1. To prove (2.1) is uniform in $\epsilon > 0$, it suffices by Proposition 2.2 to prove the uniformity for $|\tau| \leq C$ for some $C > 0$. This is the case if there are no embedded eigenvalues in $\mathbb{R}$. This happens, for example, if g is an asymptotically Euclidean scattering metric and $\partial M = \emptyset$, or if (M, g) is equal to $\mathbb{R}^n$ outside a compact set. In the latter case, for ψ satisfying (1.7),

$$\psi (P - \lambda^2)^{-1} \psi \tag{2.2}$$

continues meromorphically to

$$\lambda \in \begin{cases} \mathbb{C}, & n \text{ odd}, \\ (\mathbb{C} \setminus \{0\})^*, & n \text{ even}, \end{cases}$$

where $(\mathbb{C} \setminus \{0\})^*$ is the logarithmic Riemann surface. If, in addition, $V(x)$ satisfies (1.6), there is no pole at $\lambda = 0$, and (2.1) is uniform in $\epsilon > 0$ (see [Vai, Theorem 8, Ch. 9]).

The contours we will be using are pictured in Figures 1 and 2. For details on the meromorphic continuation, see, for example, [Sjö].

To prove Theorem 3 in general, we observe

$$\| \rho_s (P - (\tau \pm i\epsilon))^{-1} \rho_s \|_{\mathcal{H} \to \mathcal{H}} \leq \frac{C}{\epsilon}.$$

Using this estimate for $|\tau| \leq C$, we need only show (2.1) for $|\tau|$ large, which is Corollary 2.3.

It is well known (see, for example, [BrPe]) that for $R > 0$ sufficiently large, one can construct a metric $\tilde{g}$ with no trapped geodesics so that $\tilde{g}|_{X_R} = g|_{X_R}$. Let $\chi_s \in \mathcal{C}^\infty(M)$, $\operatorname{supp} \chi_s \subset X_{R+1}$, and $\chi_s(x) \equiv d_g(x, x_0)^{-s}$ for fixed x_0 and x outside a compact set. If Δ_0 is the Laplace-Beltrami operator associated to $\tilde{g}$, we have

$$\Delta_g \chi_s = \Delta_0 \chi_s,$$

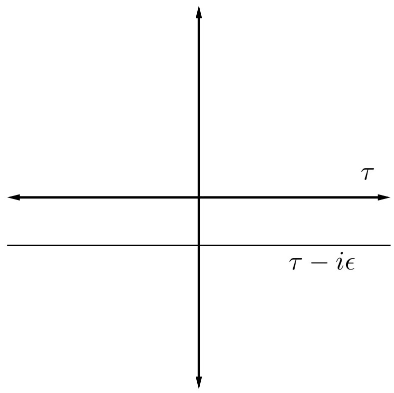

Figure 1. The curve $\tau - i\epsilon$ in the $z \in \mathbb{C}$ plane.

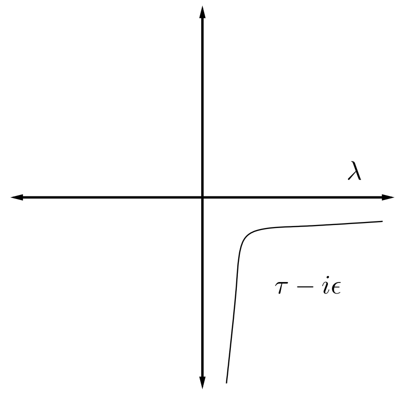

Figure 2. The same curve in the $z^{\frac{1}{2}}$ plane.

Proposition 2.3 and the Remark immediately following from [CPV] show if $s > 1/2$, and $\chi \in \mathcal{C}^\infty(M)$, $\chi \equiv 1$ on $\operatorname{supp} \chi_s$ and $\operatorname{supp} \chi \subset X_R$, then

$$\|\chi_{-s}(P(h) - E \pm i\epsilon)\chi u\|_{L^2(M)} \geq Ch\|\chi_s u\|_{H^1_h(M)} \tag{2.3}$$

for $h > 0$ sufficiently small. Here $H^1_h(M)$ is the semiclassical Sobolev space equipped with the norm

$$\|u\|^2_{H^1_h(V)} = \|u\|^2_{L^2(V)} + \|h\nabla u\|^2_{L^2(V)}.$$

We prove the presence of γ forces a weaker estimate.

Proposition 2.2. *Let (M, g) satisfy the above assumptions. For each $\rho_s \in \mathcal{C}^\infty(M)$ satisfying* (1.3) *there exist constants $C, h_0 > 0$ such that for $0 < h \leq h_0$*

$$\left\|\rho_s(P(h) - (E \pm i\epsilon))^{-1}\rho_s\right\|_{\mathcal{H}\to\mathcal{H}} \leq Ch^{-1}\log(1/h), \tag{2.4}$$

uniformly in $\epsilon > 0$.

We remark that an estimate similar to (1.10) was obtained in [BuZw] under some more assumptions, and in that work the authors implicitly suggested a result such as Proposition 2.2 should be possible.

From Proposition 2.2 we will be able to deduce the following Corollary by rescaling. We state a version both for τ and for λ.

Corollary 2.3. *Let (M, g) satisfy the above assumptions. For each $\rho_s \in \mathcal{C}^\infty(M)$ satisfying* (1.3)*, there exists a constant C such that*

$$\left\|\rho_s(-\Delta_g + V(x) - \tau)^{-1}\rho_s\right\|_{\mathcal{H}\to\mathcal{H}} \leq C\frac{\log(2 + |\tau|)}{\langle\tau\rangle^{1/2}},$$

for $|\tau| \geq C$ and

$$|\operatorname{Im}\tau| \leq \frac{\langle\tau\rangle^{1/2}}{C\log(2 + |\tau|)}.$$

Furthermore,

$$\left\|\rho_s(-\Delta_g + V(x) - \lambda^2)^{-1}\rho_s\right\|_{\mathcal{H}\to\mathcal{H}} \leq C'\frac{\log(2 + |\lambda|)}{\langle\lambda\rangle},$$

for $|\lambda| \geq C'$ *and*

$$|\operatorname{Im}\lambda| \leq \frac{1}{C' \log(2+|\lambda|)}.$$

Proof of Proposition 2.2. Observe for $\pm \operatorname{Im} z \geq c_0 h/\log(1/h)$, (2.4) holds automatically so we need only prove the Proposition for $|\operatorname{Im} z| \leq c_0 h/\log(1/h)$ for some small constant $c_0 > 0$. Let $z \in [E-\delta, E+\delta] - i(c_0 h/\log(1/h), 0)$, $\delta > 0$, for the remainder of the proof.

The idea of the proof will be to glue two cutoff resolvent estimates together and control the interaction terms by propagation of singularities, and then replace the cutoffs with ρ_s, again controlling the errors with propagation of singularities. There are 4 main steps.

Step 1: Select cutoffs.

Recall we have defined $X_r = [r, +\infty) \times S$ in the introduction, chosen $R_0 > 0$ sufficiently large so that we can construct $\widetilde{P}(h) = \operatorname{Op}(\tilde{p})$ which agrees with $P(h)$ on X_{R_0}, and the Hamiltonian flow of $\tilde{p}$ is globally non-trapping. Choose $\psi \in \mathcal{C}_c^\infty(M)$, $0 \leq \psi \leq 1$, $\psi \equiv 1$ on $M \setminus X_{R_0}$, $\psi \equiv 0$ on X_{R_0+1}, and select $\chi_0, \chi_1 \in \mathcal{C}_c^\infty(M)$, $0 \leq \chi_0 \leq 1$, $\chi_0 \equiv 1$ near γ with small support, $\chi_1 = \psi - \chi$.

In order to control the interaction (commutator) terms, we will add a complex absorption potential to $P(h) - z$ which is supported away from the above cutoffs, which will control the interactions through propagation of singularities. Choose R_j, $j = 1, \ldots, 7$,

$$R_0 + 1 =: R_1, \;\; R_j < R_{j+1} < \infty,$$

and let

$$A_{R_{j_1}, R_{j_2}} = X_{R_{j_1}} \setminus X_{R_{j_2}}$$

be the annulus with inner radius R_{j_1} and outer radius R_{j_2}. We will fix the distances between the R_js at the end of the proof.

Choose $a \in \mathcal{C}_c^\infty(M)$, $a\psi \equiv 0$,

$$a \equiv 1 \text{ on } A_{R_2,R_5}, \;\; \operatorname{supp} a \subset A_{R_1,R_6}$$

and choose $\psi_1, \psi_2 \in \mathcal{C}_c^\infty(M)$ satisfying $\psi_1 = \psi_2^2$ and

$$\operatorname{supp} \psi_2 \subset A_{R_2,R_5}, \;\; \psi_2 \equiv 1 \text{ on } A_{R_3,R_4}.$$

Set $Q(z) = P(h) - z - iC_1 h a$ for a constant $C_1 > 0$ to be chosen later in the proof.

Recall

$$\gamma \Subset U_0 \Subset U \Subset X_0,$$

and choose $\tilde{\chi} \in \mathcal{C}_c^\infty(M)$ satisfying

$$\begin{aligned} \tilde{\chi} &\equiv 1 \text{ on } M \setminus X_{R_6} \setminus U \text{ and} \\ \operatorname{supp} \tilde{\chi} &\subset M \setminus X_{R_7} \setminus U_0. \end{aligned}$$

Without loss of generality, we assume ρ_s from the statement of the Proposition satisfies (1.3) and $\rho_s \equiv 1$ on $M \setminus X_{R_7}$. These cutoffs are shown pictorially in Figure 3.

We will also employ an energy cutoff, to separate the characteristic variety of $p - E$ from the elliptic sets. Choose $\varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$,

$$\varphi_1(t) \equiv 1 \text{ on } \{|t| \leq \alpha/2\}, \;\; \varphi_1(t) \equiv 0 \text{ on } \{|t| \geq \alpha\}.$$

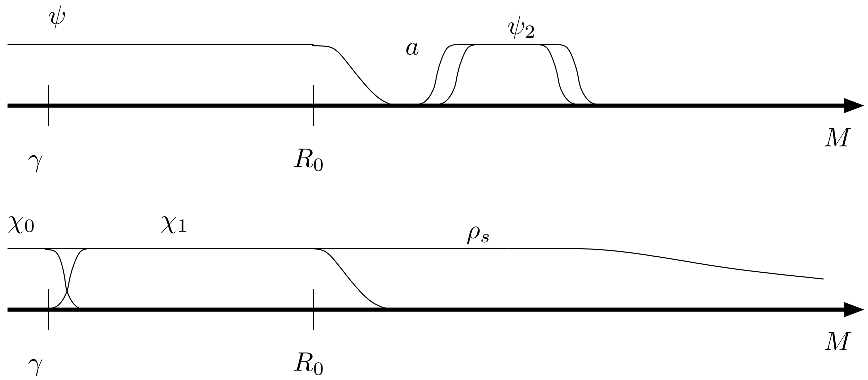

FIGURE 3. The manifold M with various cutoff functions employed in the proof of Proposition 2.2.

Set

$$\varphi(x,\xi) = \varphi_1(p(x,\xi) - E),$$

and observe since φ is a function of the principal symbol of $P(h) - z$ and we are using the Weyl calculus,

$$[P(h) - z, \varphi^w] = \mathcal{O}(h^3). \tag{2.5}$$

Step 2: Microlocalization.
We will bound $\|\psi u\|$ from above, where unless explicitly noted, $\|\cdot\| = \|\cdot\|_{\mathcal{H}}$. To do this, calculate

$$\|\psi u\| \leq \|\chi_0 u\| + \|\chi_1 u\| =: A + B.$$

For B we cutoff in energy to apply [Chr1, Theorem 1, Corollary 8] and the generalizations from [Chr2] for $c_0 > 0$ sufficiently small:

$$\begin{aligned}
B &\leq \|\varphi^w \chi_0 u\| + \|(1-\varphi)^w \chi_0 u\| \\
&\leq \frac{C\log(1/h)}{h} \|(P(h)-z)\varphi^w \chi_0 u\| + \frac{C}{\alpha} \|(P(h)-z)(1-\varphi)^w \chi_0 u\| \\
&\leq \frac{C\log(1/h)}{h} \|\varphi^w (P(h)-z)\chi_0 u\| + Ch^2 \log(1/h)\|\chi_0 u\| \qquad (2.6)\\
&\quad + \frac{C}{\alpha} \|(1-\varphi)^w (P(h)-z)\chi_0 u\| + \frac{C}{\alpha} h^3 \|\chi_0 u\| \qquad (2.7)\\
&\leq \frac{C\log(1/h)}{h} (\|Q(z)u\| + \|[P(h),\chi_0]u\|) + Ch^2\log(1/h)\|\chi_0 u\|, \qquad (2.8)
\end{aligned}$$

since $\chi a = 0$. Here in (2.6-2.7) we have used (2.5).

To estimate A and the commutator term in (2.8) we will need the lemmas in Step 3.

Step 3: Two Lemmas.
The first Lemma is a refinement of the standard propagation of singularities result.

Lemma 2.4. *Let $\widetilde{V}_1, \widetilde{V}_2 \Subset M$, and for $j = 1, 2$ let $V_j \Subset T^*M$,*

$$V_j := \{(x,\xi) \in T^*M : x \in \widetilde{V}_j,\ |p(x,\xi) - E| \leq \alpha\},$$

for some $\alpha > 0$. Suppose the $\widetilde{V}_j$ satisfy $\mathrm{dist}_g(\widetilde{V}_1, \widetilde{V}_2) = L$, and assume

$$\text{(2.9)} \qquad \begin{cases} \exists C_1, C_2 > 0 \text{ such that } \forall \rho \text{ in a neighbourhood of } V_1, \\ \exp(tH_p)(\rho) \in V_2 \text{ for} \\ \sqrt{E}(L + C_1) \leq t \leq \sqrt{E}(L + C_1 + C_2). \end{cases}$$

Suppose $A \in \Psi_h^{0,0}$ is microlocally equal to 1 in V_2. If $B \in \Psi_h^{0,0}$ and $\mathrm{WF}_h(B) \subset V_1$, then there exists a constant $C > 0$ depending only on C_1, C_2 such that

$$\begin{aligned} \|Bu\| \leq{}& CLh^{-1}\|B\|_{\mathcal{H}\to\mathcal{H}}\,\|(P(h)-z)u\| + 2(E+\alpha)^{3/4}\frac{(C_1+1)}{\sqrt{C_2}}\|B\|_{\mathcal{H}\to\mathcal{H}}\|Au\| \\ &+\mathcal{O}(h)\|\widetilde{B}u\|, \end{aligned}$$

where

$$\widetilde{B} \equiv 1 \text{ on } \cup_{0\leq t\leq\sqrt{E}(L+C_1+C_2)} \exp(tH_p)(\mathrm{WF}_h B).$$

Remark 2.5. Observe Lemma 2.4 is a statement about the principal symbol $p - z$, and hence applies also to $Q(z)$, and the difference is $\mathcal{O}(h^\infty)\|\widetilde{B}u\|$.

Proof. Let $G = \mathrm{Op}^w(g) \in \Psi_h^{0,0}$ be a self-adjoint operator to be determined later in the proof and calculate

$$\begin{aligned} \frac{L^2}{2h^2}\|G(P-z)u\|^2 + \frac{h^2}{2L^2}\|Gu\|^2 &\geq \mathrm{Im}\,\langle G(P-z)u, Gu\rangle \\ &= \mathrm{Im}\,\langle [G,P]u, Gu\rangle \\ &\geq \frac{h}{2}\langle \mathrm{Op}^w(\{p, g^2\})u, u\rangle - \mathcal{O}(h^3)\|\widetilde{G}u\|^2, \end{aligned} \tag{2.10}$$

where $\widetilde{G} \equiv 1$ on $\mathrm{WF}_h G$. Hence

$$\frac{L^2}{2h^2}\|G(P-z)u\|^2 \geq \frac{h}{2}\langle \mathrm{Op}^w(\alpha)u, u\rangle - \mathcal{O}(h^3)\|\widetilde{G}u\|^2,$$

where

$$\alpha(x,\xi) = \{p, g^2\} - \frac{h}{T^2}g^2.$$

Choose $\varphi_0 \in \mathcal{C}^\infty(M)$, $0 \leq \varphi_0 \leq 1$, satisfying

$$\begin{aligned} \varphi_0 &= \varphi_1^2, \text{ for } \varphi_1 \in \mathcal{C}^\infty(M), \\ \varphi_0 &\equiv 1 \text{ on } \widetilde{V}_1, \\ \mathrm{supp}\,|\nabla\varphi_0| &\subset \widetilde{V}_2, \text{ and} \\ |\nabla\varphi_0| &\leq \frac{2}{C_2}. \end{aligned}$$

Choose also $\varphi \in \mathcal{C}^\infty(T^*M)$, $\varphi = \varphi(p(x,\xi) - E)$ so that $\varphi^2 \equiv 1$ on $V_1 \cup V_2$. According to (2.9), we can find a non-characteristic hypersurface Σ near V_1 so that

$$V_1 \cup V_2 \Subset \bigcup_{0\leq t\leq\sqrt{E}(L+C_1+C_2)} \exp(tH_p)(\Sigma) =: \widetilde{\Sigma}.$$

Choose $f \in \mathcal{C}_c^\infty(\Sigma)$, $0 \leq f \leq 1$ so that V_1 and V_2 are contained also in the flowout of $\{f = 1\}$, and choose $\chi_0 \in \mathcal{C}_c^\infty(T^*M)$, $0 \leq \chi_0 \leq 1$, satisfying $\chi_0 \equiv 1$ on V_1 and $\chi_0 \equiv 0$ outside a neighbourhood of V_1. Let $q, a_0 \in \mathcal{C}^\infty(T^*M)$ be the solutions to

$$H_p q = \chi_0, \;\; q|_\Sigma = f,$$
$$H_p a_0 = 1, \;\; a|_\Sigma = 0.$$

Observe q satisfies

$$1 \leq q \leq \sqrt{E+\alpha} C_1 \text{ on } V_1;$$
$$|q| \leq \sqrt{E+\alpha}(C_1 + 1) \text{ on } \widetilde{\Sigma},$$

if $\operatorname{supp} \chi_0$ is sufficiently small. In addition, a_0 satisfies

$$0 \leq a_0 \leq \sqrt{E+\alpha}(L + C_1 + C_2) \text{ on } \widetilde{\Sigma}.$$

Set

$$g^2 = \varphi^2(p - E)\varphi_0(x) q^2 \exp(2ha_0/L^2),$$

so that with this choice of g^2,

$$\begin{aligned} \alpha &= \{p, g^2\} - \frac{h}{T^2} g^2 \\ &= 2q\{p,q\}e^{2ha_0/L^2} + 2\frac{h}{T^2}g^2 + q^2 e^{2ha_0/L^2}\varphi^2\{p,\varphi_0\} - \frac{h}{T^2}g^2 \\ &\geq 2q\{p,q\}e^{2ha_0/L^2} + \frac{h}{T^2}g^2 - 2(E+\alpha)^{3/2}\frac{(C_1+1)^2}{C_2} \end{aligned} \tag{2.11}$$

Combining (2.10) with (2.11) gives the lemma. □

The second Lemma will follow from Lemma 2.4 and indicates how to control the interaction terms of the form $\|[P, \chi]u\|$.

Lemma 2.6. *Suppose $\chi \in \mathcal{C}_c^\infty(M)$ satisfies*

$$\operatorname{supp} \chi \subset M \setminus X_{R_0}, \; \textit{and } \nabla\chi \equiv 0 \textit{ near } \gamma.$$

Then

$$h^{-1}\|[P(h), \chi]u\| \leq CR_7 h^{-1}\|Q(z)u\| + \mathcal{O}(h)\|\tilde{\chi}u\|,$$

where $\tilde{\chi} \in \mathcal{C}_c^\infty(M)$ was selected in Step 1.

Proof. We first microlocalize using φ^w as in Step 2. Observe $[P(h), \chi] = hA(x, hD)$, where $A(x, hD)$ is a first order semiclassical differential operator with coefficients supported in $M \setminus X_{R_0} \setminus \operatorname{neigh}(\gamma)$. We calculate

$$\|A(x,hD)u\| \leq \|A(x,hD)\varphi^w u\| + \|A(x,hD)(1-\varphi)^w u\|. \tag{2.12}$$

Now

$$\|A(x,hD)\varphi^w u\| \leq |E+\alpha| \|\nabla\chi\|_{L^\infty}\|\varphi_2^w u\|$$

for $\varphi_2 \in \mathcal{C}_c^\infty(T^*M)$, $0 \leq \varphi_2 \leq 1$, a microlocal cutoff supported away from γ. From Lemma 2.4, we have

$$\begin{aligned} &|E+\alpha| \|\nabla\chi\|_{L^\infty}\|\varphi_2^w u\| \\ &\quad \leq \; CR_7 h^{-1}\|Q(z)u\| + C\|\nabla\chi\|_{L^\infty}\|\psi_1 u\| + \mathcal{O}(h)\|\tilde{\chi}u\|. \end{aligned}$$

For the second term in (2.12) choose $\chi_2 \in \mathcal{C}_c^\infty(M)$ satisfying $\chi_2 \equiv 1$ on $\operatorname{supp} \nabla \chi$ with support in a slightly larger set, and $|\nabla \chi_2| \leq 2|\nabla \chi|$. We calculate:

$$\begin{aligned}\|A(x,hD)(1-\varphi)^w u\|^2 &= \langle A(x,hD)^* A(x,hD) \chi_2 (1-\varphi)^w u, \chi_2(1-\varphi)^w u\rangle \\ &\leq \frac{1}{2}\|A^* A \chi_2 (1-\varphi)^w u\|^2 + \frac{1}{2}\|\chi_2(1-\varphi)^w u\|^2 \\ &\leq \frac{C}{\alpha}\|(P(h)-z)\chi_2(1-\varphi)^w u\|^2 \\ &\leq \frac{C}{\alpha} R_7 \|Q(z)u\|^2 + Ch^2 \|\nabla \chi\|_{L^\infty} \|\psi_1 u\|^2 + \mathcal{O}(h^4)\|\tilde{\chi} u\|^2,\end{aligned}$$

where we have again used Lemma 2.4, (2.5) and the fact that $P(h)-z$ is a second order elliptic semiclassical differential operator on $\operatorname{supp}(1-\varphi)^w$.

We have shown

$$\begin{aligned} &h^{-1}\,\|[P(h),\chi]u\| \\ &\quad \leq CR_7 h^{-1}\|Q(z)u\| + C\|\nabla\chi\|_{L^\infty}\|\psi_1 u\| + \mathcal{O}(h)\|\tilde{\chi}u\|. \end{aligned} \tag{2.13}$$

We now use the special structure of $Q(z)$ to absorb the error terms. To do this, choose $C_1 > 0$ sufficiently large that

$$(C_1 a - c_0/\log(1/h))\psi_1 \geq (C_1 a - c_0)\psi_1 \geq c_0 \psi_1/2,$$

and recall $\psi_1 = \psi_2^2$. Then

$$\|\psi_1 u\| \leq \|\psi_2 u\|,$$

and for any $\eta > 0$,

$$\begin{aligned}\frac{1}{2} c_0 h \int |\psi_2 u|^2\, dx &\leq h \int (C_1 a + \operatorname{Im} z/h)\, u \overline{\psi_1 u} dx \\ &\leq -\operatorname{Im} \int Q(z) u \overline{\psi_1 u} dx \\ &\leq (4\eta h)^{-1}\|Q(z)u\|^2 + \eta h \|\psi_1 u\|^2.\end{aligned}$$

Combining the last two inequalities yields

$$\|\psi_1 u\|^2 \leq C(4\eta h^2)^{-1}\|Q(z)u\|^2 + C\eta\|\psi_1 u\|^2,$$

which, for sufficiently small $\eta > 0$ independent of h, gives

$$\|\psi_1 u\|^2 \leq C(4\eta h^2)^{-1}\|Q(z)u\|^2.$$

Plugging into (2.13) gives

$$h^{-1}\,\|[P(h),\chi]u\| \leq Ch^{-1}R_7\|Q(z)u\| + \mathcal{O}(h)\|\tilde{\chi}u\|.$$

□

Step 4: $(P-z)^{-1}$ and ρ_s.

We have shown

$$\|\psi u\| \leq \frac{C\log(1/h)}{h} R_7 \|Q(z)u\| + \mathcal{O}(h)\|\tilde{\chi}u\|, \tag{2.14}$$

but we have yet to replace $Q(z)$ in the estimate with $P(h)-z$ and add the weights ρ_s. Recall we have assumed $\rho_s \equiv 1$ on $\operatorname{supp}\tilde{\chi}$, and we have yet to determine the R_js. Then

$$\|\rho_s u\| \leq \|\psi u\| + \|\rho_s(1-\psi)u\|.$$

Recall there is $\widetilde{P}(h)$ which agrees with $P(h)$ on $\text{supp}\,(1-\psi)$, and the principal symbol, $\tilde{p}$, of $\widetilde{P}(h)$ has globally non-trapping classical flow. Applying [CPV, Theorem 1.1], we get

$$\begin{aligned}\|\rho_s(1-\psi)u\| &\leq Ch^{-1}\|\rho_{-s}(\widetilde{P}(h)-z)(1-\psi)u\| \\ &= Ch^{-1}\|\rho_{-s}(P(h)-z)(1-\psi)u\|.\end{aligned}$$

Thus

$$C\|\rho_{-s}(P(h)-z)u\|^2 \tag{2.15}$$
$$\begin{aligned}&\geq C_2\left(\|\rho_{-s}(1-\psi)(P(h)-z)u\|^2+\|\psi(P(h)-z)u\|^2\right) \qquad (2.16)\\ &\geq \|\rho_{-s}(P(h)-z)(1-\psi)u\|^2+\|(P(h)-z)\psi u\|^2 \\ &\quad +2\|[P(h),\psi]u\|^2 \\ &\quad -2\|[P(h),\psi]u\|\left(\|(P(h)-z)\psi u\|+\|\rho_{-s}(P(h)-z)(1-\psi)u\|\right).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the last term on the right hand side yields

$$\begin{aligned}&2\|[P(h),\psi]u\|\left(\|(P(h)-z)\psi u\|+\|\rho_{-s}(P(h)-z)(1-\psi)u\|\right) \\ &\leq 2\|[P(h),\psi]u\|(\|\psi(P(h)-z)u\|+\|\rho_{-s}(1-\psi)(P(h)-z)u\|+2\|[P(h),\psi]u\|) \\ &\leq 2\left(3\|[P(h),\psi]u\|^2+\frac{1}{2}\|\psi(P(h)-z)u\|^2+\frac{1}{2}\|\rho_{-s}(1-\psi)(P(h)-z)u\|^2\right).\end{aligned}$$

Plugging into (2.15), we have

$$\begin{aligned}C_3\|\rho_{-s}(P(h)-z)u\|^2 &\geq \|\rho_{-s}(P(h)-z)(1-\psi)u\|^2+\|(P(h)-z)\psi u\|^2 \\ &\quad -4\|[P(h),\psi]u\|^2. \qquad (2.17)\end{aligned}$$

Applying Lemma 2.4 to the last term in (2.17) with $A=\rho_s(1-\psi)$, we get

$$\begin{aligned}&\|[P(h),\psi]u\| \\ &\quad \leq C'R_7\|(P(h)-z)u\|+\frac{C}{(R_4-R_3)^{\frac{1}{2}}}h\|\rho_s(1-\psi)u\|+\mathcal{O}(h^2)\|\tilde{\chi}u\|,\end{aligned}$$

with C here independent of R_3, R_4, and h. Hence

$$\begin{aligned}C_3\|\rho_{-s}(P(h)-z)u\|^2 &\geq \|\rho_{-s}(P(h)-z)(1-\psi)u\|^2+\|(P(h)-z)\psi u\|^2 \\ &\quad -4\frac{C^2}{R_4-R_3}h^2\|\rho_s(1-\psi)u\|^2-\mathcal{O}(h^4)\|\tilde{\chi}u\|^2 \\ &\geq \frac{h^2}{C^2}\|\rho_s(1-\psi)u\|^2+\frac{h^2}{C^2\log^2(1/h)}\|\psi u\| \\ &\quad -4\frac{C^2}{R_4-R_3}h^2\|\rho_s(1-\psi)u\|^2-\mathcal{O}(h^4)\|\tilde{\chi}u\|^2 \\ &\geq \frac{h^2}{C_4\log^2(1/h)}\|\rho_s u\|^2,\end{aligned}$$

as long as $R_4-R_3>0$ is sufficiently large but fixed. Fixing the other R_js appropriately gives (2.4).

□

Theorem 3 now follows immediately from Corollary 2.3. □

3. Proof of Theorems 1 and 2

3.1. **Proof of Theorem 1.** In this section we show how to use the estimate (2.4) to prove Theorem 1. This is an adaptation of the similar proof in [Bur2], in the case M is Euclidean space with several convex bodies removed and compactly supported weights.

Let ρ_s satisfy (1.3), let $\mu = \tau \pm i\epsilon$, and suppose u and f satisfy

$$(\Delta_g - V + \mu)u = \rho_s f. \tag{3.1}$$

We multiply by $\rho_s^2 \bar{u}$ and integrate:

$$\begin{aligned} &\int \rho_s^2 \bar{u} \Delta u + \int (\mu - V)\rho_s^2 |u|^2 = \int \rho_s^3 f \bar{u} \\ &\implies -\int \rho_s^2 |\nabla u|^2 + \int \mu \rho_s^2 |u|^2 - \int (\nabla u, \nabla(\rho_s^2))\bar{u} = \int \rho_s^3 f \bar{u} \end{aligned}$$

which implies

$$\begin{aligned} \int \rho_s^2 |\nabla u|^2 \quad &\leq \quad (|\tau| + C) \int \rho_s^2 |u|^2 + \beta \int |\nabla u|^2 |\nabla(\rho_s^2)|^2 \rho_{-s}^2 \\ &\quad + (4\beta)^{-1} \int \rho_s^2 |u|^2 + |\int \rho_s^3 f \bar{u}| \end{aligned}$$

for any $\beta > 0$, since V is bounded. We observe

$$|\nabla(\rho_s^2)| \leq C \langle x \rangle^{-2s-1}$$

for large $|x|$, and hence

$$|\nabla(\rho_s^2)|^2 \rho_{-s}^2 \leq C \rho_s^2$$

for large $|x|$. This combined with $\rho_s^2 \leq C\rho_s$ implies

$$\int \rho_s^2 |\nabla u|^2 \leq (|\tau| + C)\|\rho_s u\|_{L^2}^2 + \|\rho_s f\|_{L^2}^2. \tag{3.2}$$

Now (3.1) implies

$$\begin{aligned} (|\tau| + C)^{1/2} \|\rho_s u\|_{L^2} \quad &\leq \quad (|\tau| + C)^{1/2} \|\rho_s (\Delta_g - V + \mu)^{-1} \rho_s f\|_{L^2} \\ &\leq \quad C \log(2 + |\tau|) \|f\|_{L^2}, \end{aligned}$$

which combined with (3.1) gives

$$\|\rho_s u\|_{H^1}^2 \leq C \int \rho_s^2 |\nabla u|^2 + C \int \|\rho_s u\|_{L^2}^2 \leq C \log(2 + |\tau|) \|f\|_{L^2}^2.$$

This combined with the standard interpolation arguments gives the following lemma.

Lemma 3.1. *With the notation and assumptions above, we have*

$$\|\rho_s(-\Delta_g + V - (\tau \pm i\epsilon))^{-1} \rho_s\|_{L^2 \to H^1} \leq C_\epsilon \log(2 + |\tau|)$$

and for every $\delta > 0$, $r \in [-1, 1]$,

$$\|\rho_s(-\Delta_g + V - (\tau \pm i\epsilon))^{-1} \rho_s\|_{H^r \to H^{1+r-\delta}} \leq C_{\epsilon,\delta}.$$

Now let A be the operator

$$Au_0 = \rho_s e^{-itP} u_0,$$

acting on $L^2(M)$. We want to show

$$A : L^2(M) \to L^2([0,T]; H^{\frac{1}{2}-\epsilon}(M))$$

is bounded. We use the standard argument from [BGT2]. That is, by duality, this is equivalent to the adjoint A^* being bounded

$$A^* : L^2([0,T]; H^{-\frac{1}{2}+\epsilon}(M)) \to L^2(M),$$

which is equivalent to the boundedness of

$$AA^* : L^2([0,T]; H^{-\frac{1}{2}+\epsilon}(M)) \to L^2([0,T]; H^{\frac{1}{2}-\epsilon}(M)).$$

The definition of A gives

$$A^* f = \int_0^T e^{i\tau P} \rho_s f(\tau) d\tau$$

so

$$AA^* f(t) = \int_0^T \rho_s e^{-i(t-\tau)P} \rho_s f(\tau) d\tau.$$

We show AA^* is bounded. Let u be defined by

$$u(x,t) = \int_0^T e^{-i(t-\tau)P} \rho_s f(\tau) d\tau.$$

Since we are only interested in the time interval $[0,T]$, we extend f to be 0 for $t \notin [0,T]$. We write

$$\begin{aligned} AA^* f(t) &= \int_0^t \rho_s e^{-i(t-\tau)P} \rho_s f(\tau) d\tau + \int_t^T \rho_s e^{-i(t-\tau)P} \rho_s f(\tau) d\tau \\ &=: \rho_s u_1(t) + \rho_s u_2(t), \end{aligned}$$

and calculate

$$(D_t + P) u_j = (-1)^j i \rho_s f. \tag{3.3}$$

Thus boundedness of AA^* will follow if we prove u satisfying (3.3) satisfies

$$\|\rho_s u\|_{L^2([0,T]; H^{\frac{1}{2}-\epsilon})} \leq \|f\|_{L^2([0,T]; H^{-\frac{1}{2}+\epsilon})}.$$

Replacing $\pm i f$ with f in equation (3.3) and taking the Fourier transform in time results in the following equation for $\hat{u}$ and $\hat{f}$:

$$(-z + P)\hat{u}(z,\cdot) = \rho_s \hat{f}(z,\cdot). \tag{3.4}$$

Since $f(t,\cdot)$ is supported only in $[0,T]$, $\hat{f}(z,\cdot)$ and $\hat{u}(z,\cdot)$ are holomorphic, bounded, and satisfy (3.4) in $\{\operatorname{Im} z < 0\}$. Let $z = \tau - i\eta$, $\eta > 0$ sufficiently small. We apply Lemma 3.1 to get

$$\|\rho_s \hat{u}(z,\cdot)\|_{H^{\frac{1}{2}-\epsilon}(M)} \leq C \|\hat{f}(z,\cdot)\|_{H^{-\frac{1}{2}+\epsilon}(M)},$$

for $\epsilon > 0$. Thus

$$\begin{aligned}
\|\rho_s u\|_{L^2([0,T];H^{\frac{1}{2}-\epsilon}(M))} &\leq e^{\eta T}\|e^{-\eta t}\rho_s u(t)\|_{L^2([0,T];H^{\frac{1}{2}-\epsilon}(M))} \\
&\leq Ce^{\eta T}\|\rho_s \hat{u}(\tau - i\eta)\|_{L^2(\mathbb{R};H^{\frac{1}{2}-\epsilon}(M))} \\
&\leq Ce^{\eta T}\|\hat{f}(\tau - i\eta)\|_{L^2(\mathbb{R};H^{-\frac{1}{2}+\epsilon}(M))} \\
&\leq Ce^{\eta T}\|e^{-\eta t}f(t)\|_{L^2([0,T];H^{-\frac{1}{2}+\epsilon}(M))} \\
&\leq Ce^{\eta T}\|f(t)\|_{L^2([0,T];H^{-\frac{1}{2}+\epsilon}(M))}.
\end{aligned}$$

Hence

$$\int_0^T \|\rho_s u\|^2_{H^{\frac{1}{2}-\epsilon}(M)} dt \leq Ce^{\eta T}\int_0^T \|f\|^2_{H^{-\frac{1}{2}+\epsilon}(M)} dt,$$

or AA^* is bounded. Thus A is bounded and Theorem 1 is proved. □

Remark 3.2. If the estimate (2.1) is uniform in the lower half-plane, then the preceding calculation can be made including taking the limit $\eta \to 0$, in which case we get the global in time local smoothing estimate (1.4)

The following Lemma uses interpolation to replace the $H^{1/2-\epsilon}$ norm on the left hand side of (1.2) with $H^{1/2}$, and will be of use in §4.

Lemma 3.3. *Suppose (M, g) and V satisfy the assumptions of Theorem 1. For each $\delta > 0$ there is a constant $C > 0$ such that*

$$\int_0^T \left\|\rho_s e^{it(\Delta_g - V(x))}u_0\right\|^2_{H^{1/2}(M)} dt \leq C\|u_0\|^2_{H^\delta(M)}. \tag{3.5}$$

Proof. We first calculate

$$\begin{aligned}
\|\rho_s e^{itP}u_0\|^2_{L^2_T H^1} \leq\ & C\int_0^T\int_M \left|\rho_s(P+1)e^{itP}u_0\rho_s e^{-itP}\overline{u_0}\right| dxdt \\
&+2\int_0^T\int_M |\nabla\rho_s|\left|\nabla e^{itP}u_0\right|\left|\rho_s e^{itP}u_0\right| dxdt \\
&+\int_0^T\int_M |P(\rho_s)|\left|e^{itP}u_0\right|\left|\rho_s e^{itP}u_0\right| dxdt.
\end{aligned}$$

Using

$$|P(\rho_s)| \leq C|\nabla\rho_s| \leq C'|\rho_s|$$

and applying the Cauchy-Schwarz inequality yields

$$\|\rho_s e^{itP}u_0\|^2_{L^2_T H^1} \leq C\|u_0\|_{H^2}.$$

Thus we have a linear operator bounded between complex interpolation spaces:

$$\begin{aligned}
\rho_s e^{itP} &: L^2 \to L^2_T H^{1/2-\epsilon}, \\
&: H^2 \to L^2_T H^1.
\end{aligned}$$

Choosing $\epsilon = \delta/4$ we have

$$\begin{aligned}
\|\rho_s e^{itP}u_0\|_{L^2_T H^{1/2}} &\leq C\|u_0\|_{H^{2\epsilon/(1/2+\epsilon)}} \\
&\leq C\|u_0\|_{H^\delta}.
\end{aligned}$$

□

3.2. **Proof of Theorem 2.** For the proof of Theorem 2, we apply [Chr3, Theorem 3], which is a generalization of [Bur1, Théorème 3]. That is, we set

$$B = \begin{pmatrix} 0 & -i\,\mathrm{id} \\ -i\Delta & 0 \end{pmatrix},$$

acting on the Hilbert space $H = H^1(X) \times L^2(X)$. The commutator $[\psi, B]$ is bounded on H, so if $\psi_2 \in \mathcal{C}^\infty(X)$ satisfies (1.7) and $|[\psi, -\Delta_g]| \leq \psi_2$, we have

$$\begin{aligned} \|\psi e^{itB}\psi\|_{\mathrm{Dom}\,(B^2)\to H} &= \|\psi e^{itB}\psi(1-iB)^{-2}\|_{H\to H} \\ &\leq C\|\psi e^{itB}(1-iB)^{-2}\psi_2\|_{H\to H}. \end{aligned}$$

From [Chr3, Theorem 3] we then gather

$$\|\psi e^{itB}\psi\|_{\mathrm{Dom}\,(B^2)\to H} \leq Ce^{-t^{1/2}/C}.$$

The spaces $H^{1+s} \times H^s$ are complex interpolation spaces, so together with the trivial estimate

$$\|\psi e^{itB}\psi\|_{H\to H} \leq C,$$

we conclude that for any $\epsilon > 0$,

$$E_\psi(t) \leq C_\epsilon e^{-\epsilon t^{1/2}/C} \left(\|u_0\|^2_{H^{1+\epsilon}} + \|u_1\|^2_{H^\epsilon} \right).$$

□

4. Strichartz-type Inequalities

In this section we prove several families of Strichartz-type inequalities and prove Proposition 1.3. The statements and proofs are mostly adaptations of similar inequalities in [BGT2], so we leave out the proofs of these in the interest of space.

As in the statement of Proposition 1.3, we assume M is asymptotically conic as defined in [HTW] and $V \in \mathcal{C}_c^\infty(M)$, $V \geq 0$. The manifold M admits the Sobolev embeddings recorded in the following proposition. For our notation, let

$$W^{m,p}(M) \text{ (resp. } W_0^{m,p}(M)),\ m \in \mathbb{N}$$

be the completion of $\mathcal{C}^\infty(M)$ (resp. $\mathcal{C}_c^\infty(M)$) with respect to the norm

$$\|f\|^p_{W^{m,p}} = \sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p}.$$

We define $W^{s,p}(M)$ and $W_0^{s,p}(M)$ for non-integer s by interpolation. We use the convention

$$H^s(M) := W^{s,2}(M), \text{ and } H_0^s(M) := W_0^{s,2}(M).$$

Let $H_D^1(M)$ denote the domain of $(1-\Delta_g)^{\frac{1}{2}}$ with Dirichlet boundary conditions if $\partial M \neq \emptyset$, so that $H_D^1(M) = H_0^1(M)$, and write $H_D^s(M)$ for the domain of $(1-\Delta_g)^{s/2}$ with Dirichlet boundary conditions. Since $V \in \mathcal{C}_c^\infty(M)$, we may replace $(1-\Delta_g)^{\frac{1}{2}}$ with $(1+P)^{\frac{1}{2}}$ in the definitions, where $P = -\Delta_g + V(x)$. This results in equivalent Sobolev spaces with the addition $[P, (1+P)^{\frac{1}{2}}] = 0$.

Proposition 4.1. *We have the following continuous Sobolev embeddings:*

$$\text{(i) } H_D^1(M) \subset L^p(M),\ 2 \le p \le \frac{2n}{n-2},\ \text{or } p < \infty \text{ for } n = 2,$$

$$\text{(ii) } H_D^s(M) \subset L^p(M),\ \frac{1}{2} = \frac{s}{n} + \frac{1}{p},\ s \in [0,1),$$

$$\text{(iii) } H_D^{s+1}(M) \subset W^{1,p}(M),\ \frac{1}{2} = \frac{s}{n} + \frac{1}{p},\ s \in [0,1),$$

$$\text{(iv) } W_0^{1,p}(M) \subset L^q(M),\ \frac{1}{p} = \frac{1}{n} + \frac{1}{q},\ 1 \le p < q < +\infty,$$

$$\text{(v) } W_0^{s,p}(M) \subset L^\infty(M),\ s > \frac{n}{p},\ p \ge 1$$

$$\text{(vi) } H_D^{s+1/p}(M) \subset W^{s,q}(M),\ \frac{1}{p} + \frac{n}{q} = \frac{n}{2},\ p \ge 2, s \in [0,1].$$

If we again let $-\Delta_0$ be the Laplace-Beltrami operator associated to a non-trapping metric which agrees with g on X_R, we may apply the results of [HTW] to a solution of the Schrödinger equation away from the trapping region, resulting in perfect Strichartz estimates, but we lose something from the commutator. That is, if $\chi \in \mathcal{C}_c^\infty(M)$ is 1 on $M \setminus (X_R \cup \operatorname{supp} V)$, then $w = (1-\chi)e^{-itP}u_0$ satisfies

$$\begin{aligned} (D_t - \Delta_0)w &= (D_t + P)w \\ &= [\Delta_0, \chi]e^{-itP}u_0. \end{aligned} \tag{4.1}$$

From Lemma 3.3, we have for any $\epsilon > 0$,

$$\|[\Delta_0, \chi]e^{-itP}u_0\|_{L^2([0,T])H^{-1/2}(M)} \le C_{\epsilon,T}\|u_0\|_{H^\epsilon(M)}.$$

The following proposition then follows from the proof of [BGT2, Proposition 2.10].

Proposition 4.2. *For every $0 < T \le 1$, $\delta > 0$, and each $\chi \in \mathcal{C}_c^\infty(M)$ satisfying $\chi \equiv 1$ near $M \setminus (X_R \cup \operatorname{supp} V)$, there is a constant $C > 0$ such that*

$$\|(1-\chi)u\|_{L^p([0,T])W^{s-\delta,q}(M)} \le C\|u_0\|_{H_D^s(M)}, \tag{4.2}$$

where $u = e^{-itP}u_0$, $s \in [0,1]$, and (p,q), $p > 2$ satisfy

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Remark 4.3. In the sequel, wherever unambiguous, we will write

$$L_T^p W^{s,q} := L^p([0,T])W^{s,q}(M)$$

and

$$H_D^s := H_D^s(M).$$

Proposition 4.4. *Let $u(t) = e^{-itP}u_0$. For every $0 < T \le 1$ and $\epsilon > 0$, there is a constant $C > 0$ such that*

$$\|u\|_{L_T^p W^{s,q}} \le C\|u_0\|_{H_D^{s+1/p+\epsilon}}, \tag{4.3}$$

where $s \in [0,1]$ and (p,q), $p > 2$ satisfy

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}.$$

Remark 4.5. Proposition 4.4 represents the Strichartz estimates obtained by Burq-Gérard-Tzvetkov [BGT2] in the case of non-trapping exterior domains with an $\epsilon > 0$ loss due to the presence of the trapped orbit γ. Observe that (4.4) is weaker than the standard Euclidean Strichartz estimates in two ways, the loss of $1/p$ derivatives from using Sobolev embeddings and the loss of ϵ derivatives from γ. When $\partial M = \emptyset$, we get the improvement given in Proposition 4.9.

Proposition 4.6. *Let* $u = e^{-itP}u_0$ *and*

$$v = \int_0^t e^{-i(t-\tau)P} f(\tau) d\tau.$$

Then for each $0 < T \leq 1$ *and each* $\delta > 0$*, there exists* $C > 0$ *such that*

$$\|u\|_{L^p_T W^{s-\delta,q}} \leq C \|u_0\|_{H^s_D} \tag{4.4}$$

and

$$\|v\|_{L^p_T W^{s-\delta,q}} \leq C \|f\|_{L^1_T H^s_D}, \tag{4.5}$$

where $s \in [0,1]$ *and* (p,q)*,* $p > 2$ *satisfy*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2}. \tag{4.6}$$

Remark 4.7. Proposition 4.6 is much weaker than the estimate suggested by scaling in Euclidean space, and as remarked in [BGT2], is probably not optimal. We expect the $\delta > 0$ loss to always hold due to the presence of γ, but the Euclidean scaling suggests the optimal estimate would replace $1/p$ in (4.6) with $2/p$ (see Proposition 4.9).

Proposition 4.8. *Let*

$$v(t) = \int_0^t e^{-i(t-\tau)P} f(\tau) d\tau.$$

For each $0 < T \leq 1$ *and each* $\delta > 0$*, there is a constant* $C > 0$ *such that*

$$\|v\|_{L^p_T W^{s-\delta,q}} \leq C \|f\|_{L^{p'}_T W^{s,q'}}, \tag{4.7}$$

where p', q'*,* $p' \in [1,2)$ *are the duals of* p *and* q *satisfying* (4.6)*, respectively, and satisfy*

$$\frac{1}{p'} + \frac{n}{q'} = \frac{n}{2} + 1.$$

The next proposition is an improvement of Proposition 4.6 in the case $\partial M = \emptyset$.

Proposition 4.9. *Suppose* (M,g) *and* V *satisfy the assumptions of Proposition 1.3,* $u = e^{-itP}u_0$*,*

$$v = \int_0^t e^{-i(t-\tau)P} f(\tau) d\tau,$$

and in addition $\partial M = \emptyset$*. Then for each* $0 < T \leq 1$ *and each* $\delta > 0$*, we have the estimates* (4.4) *and* (4.5) *for* $s \in [0,1]$*, where now* (p,q)*,* $p > 2$ *satisfy the Euclidean scaling*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \tag{4.8}$$

Proof. The idea of the proof is to use Proposition 4.2 to reduce the statement to a local question near the trapped orbit. Then we use a partition of unity and the local WKB construction from [BGT1] to get local in time Strichartz estimates for time on the scale of inverse frequency. We then sum up over frequencies and apply the local smoothing estimate to prove the Proposition. We remark this would also follow from [StTa, Theorem 4] and the local smoothing in Theorem 1.

Let χ be as in Proposition 4.2 and choose $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\psi \equiv 1$ near 1 and satisfying

$$1 \leq \sum_{k \geq 0} \psi(r/2^k) \leq 2 \text{ for } r \geq 0.$$

Choose also $\varphi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\varphi \equiv 1$ on $[-c_0, c_0]$, $\operatorname{supp} \varphi \subset [-2c_0, 2c_0]$ for $c_0 > 0$ small. Let

$$w_h = \varphi(t/h)\chi(x)\psi(-h^2\Delta_g + h^2 V(x))u,$$

which satisfies the equation

$$\begin{cases} (i\partial_t + \Delta_g - V(x))w_h = \varphi[\Delta_g, \chi]\psi u + i\frac{1}{h}\varphi' \chi \psi u \\ w_h(x, 0) = \chi \psi u_0. \end{cases}$$

Since φ localizes to a timescale of size h, the semiclassical local WKB construction in [BGT1] gives

$$\|w_h\|_{L^p L^q} \leq C \left\| \varphi[\Delta_g, \chi]\psi u + i\frac{1}{h}\varphi' \chi \psi u \right\|_{L^1 L^2},$$

with (p, q), $p > 2$ satisfying (4.8).

Choose $\tilde{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R})$ and $\tilde{\chi} \in \mathcal{C}_c^\infty(M)$ satisfying $\tilde{\varphi} \equiv 1$ on $\operatorname{supp} \varphi$ and $\tilde{\chi} \equiv 1$ on $\operatorname{supp} \chi$. Applying Hölder's inequality in time and exchanging one derivative for h^{-1} on the support of ψ yields

$$\|w_h\|_{L^p L^q} \leq C h^{1/2} \left\| h^{-1} \tilde{\varphi} \tilde{\chi} \psi u \right\|_{L^2 L^2}.$$

Exchanging one half derivative with $h^{-1/2}$ we obtain

$$\|w_h\|_{L^p L^q} \leq C \left\| \tilde{\varphi} \tilde{\chi} \psi u \right\|_{L^2 H^{1/2}},$$

and summing in $h = 2^{-k/2}$ we get

$$\|\chi u\|_{L^p L^q} \leq C \|\tilde{\chi} u\|_{L^2 H^{1/2}},$$

which after a time truncation and an application of Lemma 3.3 proves the Proposition for u. Finally, an application of the Christ-Kiselev lemma [ChKi] proves the proposition for v. □

Proof of Proposition 1.3. The proof of Proposition 1.3 is a slight modification of the proof of Proposition 3.1 in [BGT1], but we include it here in the interest of completeness. First we assume $\partial M \neq \emptyset$. Fix s satisfying 1.12 and choose $p > \max\{2\beta - 2, 2\}$ satisfying

$$s > \frac{n}{2} - \frac{1}{p} + \delta \geq \frac{n}{2} - \frac{1}{\max\{2\beta - 2, 2\}}$$

for some $\delta > 0$. Set $\sigma = s - \delta$ and

$$Y_T = C([-T, T]; H_D^s(M)) \cap L^p([-T, T]; W_0^{\sigma, q}(M))$$

for

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2},$$

equipped with the norm

$$\|u\|_{Y_T} = \max_{|t| \leq T} \|u(t)\|_{H^s_D(M)} + \|u\|_{L^p_T W^{\sigma,q}}.$$

Let Φ be the nonlinear functional

$$\Phi(u) = e^{-itP} u_0 - i \int_0^t e^{-i(t-\tau)P} F(u(\tau)) d\tau.$$

If we can show that $\Phi : Y_T \to Y_T$ and is a contraction on a ball in Y_T centered at 0 for sufficiently small $T > 0$, this will prove the first assertion of the Proposition, along with the Sobolev embedding

$$W^{\sigma,q}_0(M) \subset L^\infty(M), \tag{4.9}$$

since $\sigma > n/q$. From Proposition 4.6, we bound the W^σ part of the Y_T norm by the H^s_D norm, giving

$$\begin{aligned} \|\Phi(u)\|_{Y_T} &\leq C \left(\|u_0\|_{H^s_D} + \int_{-T}^T \|F(u(\tau))\|_{H^s_D} d\tau \right) \\ &\leq C \left(\|u_0\|_{H^s_D} + \int_{-T}^T \|(1 + |u(\tau)|)\|^{2\beta-2}_{L^\infty}) \|u(\tau)\|_{H^s_D} d\tau \right), \end{aligned}$$

where the last inequality follows by our assumptions on the structure of F. Applying Hölder's inequality in time with $\tilde{p} = p/(2\beta - 2)$ and

$$\frac{1}{\tilde{q}} + \frac{1}{\tilde{p}} = 1$$

gives

$$\|\Phi(u)\|_{Y_T} \leq C \left(\|u_0\|_{H^s_D} + T^\gamma \|u\|_{L^\infty_T H^s_D} \|(1 + |u|)\|^{2\beta-2}_{L^p_T L^\infty}) \right)$$

where $\gamma = 1/\tilde{q} > 0$. Thus

$$\|\Phi(u)\|_{Y_T} \leq C \left(\|u_0\|_{H^s_D} + T^\gamma (\|u\|_{Y_T} + \|u\|^{2\beta}_{Y_T}) \right).$$

Similarly, we have for $u, v \in Y_T$,

$$\|\Phi(u) - \Phi(v)\|_{Y_T} \leq \tag{4.10}$$

$$\leq CT^\gamma \|u - v\|_{L^\infty_T H^s_D} \|(1 + |u|)\|^{2\beta-2}_{L^p_T L^\infty} + \|(1 + |v|)\|^{2\beta-2}_{L^p_T L^\infty}) \tag{4.11}$$

$$\leq CT^\gamma \|u - v\|_{Y_T} \|(1 + |u|)\|^{2\beta-2}_{Y_T} + \|(1 + |v|)\|^{2\beta-2}_{Y_T}),$$

which is a contraction for sufficiently small T. This concludes the proof of the first assertion in the Proposition.

To get the second assertion, we observe from 4.10 and the definition of Y_T, if u and v are two solutions to (1.11) with initial data u_0 and u_1 respectively, so

$$\widetilde{\Phi}(v) = e^{-itP} u_1 - i \int_0^t e^{-i(t-\tau)P} F(v(\tau)) d\tau,$$

we have

$$
\begin{aligned}
&\max_{|t|\le T}\|u(t)-v(t)\|_{H^s_D}\\
&\quad= \max_{|t|\le T}\|\Phi(u)(t)-\widetilde{\Phi}(v)(t)\|_{H^s_D}\\
&\quad\le C\Bigg(\|u_0-u_1\|_{H^s_D}\\
&\qquad\qquad +T^\gamma \max_{|t|\le T}\|u(t)-v(t)\|_{H^s_D}\|(1+|u|)\|^{2\beta-2}_{L^p_T L^\infty}+\|(1+|v|)\|^{2\beta-2}_{L^p_T L^\infty})\Bigg),
\end{aligned}
$$

which, for $T>0$ sufficiently small gives the Lipschitz continuity.

In the case $\partial M=\emptyset$, we have the improved Strichartz estimates given in Proposition 4.9. Hence we can take s and p satisfying $p>\max\{2\beta-2,2\}$ and

$$
s>\frac{n}{2}-\frac{2}{p}+\delta\ge\frac{n}{2}-\frac{2}{\max\{2\beta-2,2\}}
$$

for $\delta>0$. Then $\sigma=s-\delta>q/n$ and the preceding argument holds with these modifications. □

The proof of Corollary 1.4 now follows from the standard global well-posedness arguments from, for example, [Caz, Chapter 6].

□

References

[BrPe] Bruneau, V. and Petkov, V. Semiclassical Resolvent Estimates for Trapping Perturbations. *Comm. Math. Phys.* **213**, 2000, p. 413-432.

[Bur1] Burq, N. Décroissance de l'énergie locale de l'équation des ondes pour le problème extérieur et absence de résonance au voisinage du réel. *Acta Math.* **180**, 1998, p. 1-29.

[Bur2] Burq, N. Smoothing Effect for Schrödinger Boundary Value Problems. *Duke Math. Journal.* **123**, No. 2, 2004, p. 403-427.

[BGT1] Burq, N. Gérard, P., and Tzvetkov, N. Strichartz Inequalities and the Nonlinear Schrödinger Equation on Compact Manifolds. *Amer. J. Math.* **126**, No. 3, 2004, p. 569-605.

[BGT2] Burq, N., Gérard, P., and Tzvetkov, N. On Nonlinear Schrödinger Equations in Exterior Domains. *Ann. I H. Poincaré.* **21**, 2004, p. 295-318.

[BuZw] Burq, N. and Zworski, M. Geometric Control in the Presence of a Black Box. *J. Amer. Math. Soc.* **17**, 2004, p. 443-471.

[CPV] Cardoso, F., Popov, G., and Vodev, G. Semi-classical Resolvent Estimates for the Schrödinger Operator on Non-compact Complete Riemannian Manifolds. *Bull. Braz. Math Soc.* **35**, No. 3, 2004, p. 333-344.

[Caz] Cazenave, T. *Semilinear Schrödinger Equations.* Courant Lecture Notes in Mathematics, AMS, 2003.

[ChKi] Christ, M. and Kiselev, A. Maximal Functions Associated to Fiiltrations. *J. Funct. Anal.* **179**, No. 2, 2001, p. 409-425.

[Chr1] Christianson, H. Semiclassical Non-concentration near Hyperbolic Orbits. *J. Funct. Analysis*, **262** No. 2, 2007, p. 145-195.

[Chr2] Christianson, H. Quantum Monodromy and Non-concentration near Semi-hyperbolic Orbits. *in preparation.*

[Chr3] Christianson, H. Applications of Cutoff Resolvent Estimates to the Wave Equation. *preprint.* 2007.
`http://math.mit.edu/~hans/wave2-2.pdf`

[Doi] Doi, S.-I. Smoothing effects of Schrödinger Evolution Groups on Riemannian Manifolds. *Duke Mathematical Journal.* **82**, No. 3, 1996, p. 679-706.

[EvZw] Evans, L.C. and Zworski, M. *Lectures on Semiclassical Analysis.* http://math.berkeley.edu/∼evans/semiclassical.pdf.

[HTW] Hassell, A., Tao, T., and Wunsch, J. Sharp Strichartz Estimates on Non-trapping Asymptotically Conic Manifolds. *preprint.* 2004. http://www.arxiv.org/pdf/math.AP/0408273.

[Hit] Hitrik, M. Eigenfrequencies and Expansions for Damped Wave Equations. *Methods and Applications of Analysis.* **10**, No. 4, 2003, p. 543-564.

[Leb] Lebeau, G. Equation des Ondes Amorties. *Algebraic and Geometric Methods in Mathematical Physics*, A Boutet de Monvel and V. Marchenko (eds.) Kluwer Academic Publishers, Netherlands, 1996. 73-109.

[Mel] Melrose, R. Spectral and scattering theory for the Laplacian on asymptotically Euclidian spaces. *Spectral and scattering theory*, p. 85-130, *Lecture Notes in Pure and Appl. Math.* **161**, Dekker, New York, 1994.

[MeSo1] Metcalfe, J. and Sogge, C. Global existence for Dirichlet-wave equations with quadratic nonlinearities in high dimensions. *Math. Ann.* **336** (2006) p. 391-420.

[MeSo2] Metcalfe, J. and Sogge, C. Hyperbolic trapped rays and global existence of quasilinear wave equations. *Invent. Math.* **159** (2005), No. 1, p. 75-117.

[Mor] Morawetz, C. The decay of solutions of the exterior initial-boundary value problem for the wave equation. *Comm. Pure Appl. Math.* **14** (1961) p. 561-568.

[MoPh] Morawetz, C. and Phillips, R. The exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle. *Bull. Amer. Math. Soc.* **68** (1962) p. 593-595.

[MRS] Morawetz, C., Ralston, J., and Strauss, W. Decay of solutions of the wave equation outside nontrapping obstacles. *Comm. Pure Appl. Math.* **30**, No. 4, (1977) p. 447-508.

[PeZw] Petkov, V. and Zworski, M. Semi-classical Estimates on the Scattering Determinant. *Ann. I. H. Poincaré.* **2**, 2001, p. 675-711.

[Ral] Ralson, J. Trapped rays in spherically symmetric media and poles of the scattering matrix. *Comm. Pure Appl. Math.* **24** (1971), p. 571-582.

[RT] Rauch, J. and Taylor, M. Decay of Solutions to Nondissipative Hyperbolic Systems on Compact Manifolds. *Comm. Pure Appl. Math.* **28**, No. 4, 1975, p. 501-523.

[Sjö] Sjöstrand, J. *Lectures on Resonances.* http://www.math.polytechnique.fr/∼sjoestrand/CoursgbgWeb.pdf.

[StTa] Staffilani, G. and Tataru, D. Strichartz Estimates for a Schrödinger Operator with Nonsmooth Coefficients. *Comm. Partial Differential Equations.* **27**, No. 5-6, 2002, p. 1337-1372.

[Vai] Vainberg, B. *Asymptotic methods in equations of mathematical physics.* Translated from the Russian by E. Primrose. Gordon & Breach Science Publishers, New York, 1989.

[Vod] Vodev, G. Local Energy Decay of Solutions to the Wave Equation for Nontrapping Metrics. *Ark. Mat.* **42**, 2004, p. 379-397.

Department of Mathematics, MIT, 77 Mass. Ave., Cambridge, MA 02139-4307, USA
E-mail address: hans@math.mit.edu