THE COLIN DE VERDIÈRE GRAPH PARAMETER FOR THRESHOLD GRAPHS

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ABSTRACT. We consider Schrödinger operators on threshold graphs and prove a formula for the Colin de Verdière parameter in terms of the building sequence. We construct an optimal Colin de Verdière matrix for each connected threshold graph G of n vertices. For a large subclass of threshold graphs we construct an alternative Colin de Verdière matrix depending on a large parameter. As a corollary to this last construction, we give estimates on the size of the non-zero eigenvalues of this matrix.

1. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we consider Schrödinger operators on graphs with a weight or metric on the edges. We present a formula for the Colin de Verdière graph parameter for threshold graphs and show that the proof of the formula provides an algorithm for constructing an optimal Colin de Verdière matrix. Finally, motivated by [ChRe] we exploit the special structure of the (flat) graph Laplacian for threshold graphs to provide an alternate construction of an optimal Colin de Verdière matrix for a large subclass of threshold graphs.

The Colin de Verdière number $\mu(G)$ of a graph is, roughly speaking, a measure of the geometric complexity of a graph. It has been introduced in [CdV]. The comprehensive survey [HLS] is a reference for most of the known facts about $\mu(G)$.

Our first result is the following theorem.

Theorem 1. *Let G be a connected threshold graph on n vertices built by adding i isolate vertices and c cone vertices in some order.*

1. *If the construction sequence of G is of the form "cone, cone, ...", then $\mu(G) = c - 1$.*
2. *If the construction sequence of G is of the form "cone, isolate, cone, ...", then $\mu(G) = c - 1$.*
3. *If the construction sequence of G is of the form "cone, isolate, isolate, ...", then $\mu(G) = c$.*

Furthermore, an optimal Colin de Verdiére matrix for G can be produced.

We say a graph G is *threshold* if it is built inductively from a single vertex by adding vertices one at a time according to the following rules:

(1) Either make an edge from the new vertex to all previous vertices, or

(2) Make no new edges.

In case (1) we say the new vertex is a *cone*, and in case (2) the new vertex is an *isolate*. By convention in this work, we always refer to the first vertex as a cone. Hence a threshold graph might have building sequence

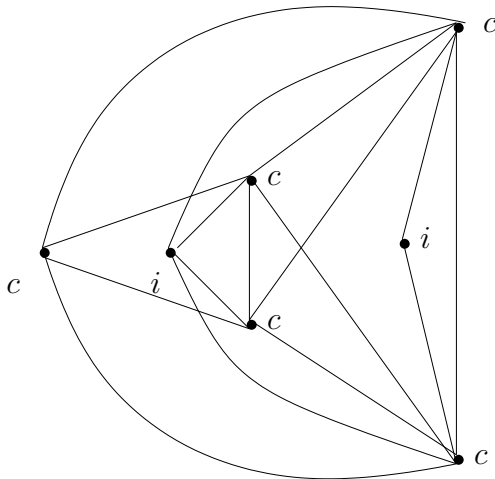cone, isolate, cone, cone, isolate, cone, cone.

FIGURE 1. A threshold graph built left to right.

This graph is pictured in Figure 1. Clearly a threshold graph is connected if and only if the building sequence ends with a cone.

Any two cones which appear sequentially in the building sequence are adjacent to the same vertex set, hence are equivalent up to graph isomorphism, and similarly for isolates which appear sequentially in the building sequence. Hence we may phrase the construction in terms of *blocks* of cones or isolates which appear sequentially. We obtain a *block sequence*

$$k_1, i_1, \ldots, k_m, i_m, k_{m+1}$$

which begins with a block of cones and ends with a block of cones under the assumption that G is connected. The block sequence for the example in Figure 1 is

$$1, 1, 2, 1, 2.$$

Every vertex in each block has the same degree, with the last cones added having the largest degree $d_1 = n - 1$ and the last isolates having the smallest degree $d_{2m+1} = k_{m+1}$.

There are also equivalent definitions of threshold graphs - we summarize here some of them:

Theorem 2 ([ChHa]). *The following statements are equivalent for a graph G:*

(1) *G is a threshold graph.*
(2) *There exist weights $w_v \geq 0$ and a number t so that for all pairs of vertices $u \neq v$ it holds that $w_u + w_v > t$ if and only if u and v are adjacent.*
(3) *G does not contain P_4, C_4 or $2K_2$ as an induced subgraph.*
(4) *There is an assignment of weights $w_v \geq 0$ to the vertices and a number $t \geq 0$ so that for a set of vertices X it holds that $\sum_{v \in X} w_v \leq t$ if and only if X is independent.*

Number the vertices of a graph G in order of weakly decreasing degree, $v_1, \ldots, v_n$. We define the graph *Laplacian* or *incidence matrix* to be the matrix $L(G)$ given by

$$L(G)_{ij} = \begin{cases} -\#\{\text{edges between } v_i \text{ and } v_j\}, & i \neq j \\ \text{degree of } v_i, & i = j. \end{cases}$$

For the example of Figure 1, we have

$$L(G) = \begin{pmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 5 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 5 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

For a less trivial example, the block sequence

$$(1.1) \qquad\qquad\qquad 2, 2, 1, 1, 3, 2, 1$$

results in the Laplacian

$$(1.2)\ L(G) =$$

$$\begin{pmatrix} 11 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 9 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 8 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 6 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 6 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 5 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The structure of this matrix arising from the assumption that G be a connected threshold graph is apparent in this example. Most notably, the matrix can be given by row blocks or column blocks corresponding to the block sequence, and the row blocks for cones are characterized by having no zeros before the diagonal. We will use this in §3.

Next we define *Colin de Verdière (CdV)* matrices, which are edge-weighted incidence matrices plus a vertex potential satisfying some non-degeneracy assumptions. Specifically, we have the following definition from [HLS]: A symmetric, real-valued $n \times n$ matrix M is a CdV matrix if

$(1.3) \qquad \bullet i \neq j \implies M_{ij} < 0$ if v_i and v_j are adjacent and $M_{ij} = 0$ if not;

$(1.4) \qquad \bullet M$ has exactly one negative eigenvalue of multiplicity 1;

$(1.5) \qquad \bullet$ there is no non-zero symmetric matrix X satisfying $MX = 0$

$$\text{and } X_{ij} = 0 \text{ if } i = j \text{ or } M_{ij} \neq 0.$$

We think of the hypothesis (1.3) as saying M is a Schrödinger operator on G, $M = L_g(G) + V(G)$, where L_g is the graph Laplacian in some Riemannian (edge-weighted) metric g and $V(G)$ is a graph potential giving weight to the vertices.

The assumptions (1.4-1.5) correspond roughly to saying the metric and potential are non-degenerate in some sense.

For the example of Figure 1, the construction in the proof of Theorem 1 yields:

$M =$

$$(1.6) \quad \begin{pmatrix} -0.471 & -0.555 & -1.02 & -1.02 & -0.721 & -0.721 & -1 \\ -0.555 & -0.302 & -0.474 & -0.474 & -0.335 & -0.335 & -0.129 \\ -1.02 & -0.474 & 0 & -0.707 & -1 & -1 & 0 \\ -1.02 & -0.474 & -0.707 & -0.707 & -0.5 & -0.5 & 0 \\ -0.721 & -0.335 & -1 & -0.5 & 0 & 0 & 0 \\ -0.721 & -0.335 & -1 & -0.5 & 0 & 0 & 0 \\ -1 & -0.129 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

while the alternative construction in §3 gives

$$(1.7) \quad M = \begin{pmatrix} -a & -a & -b & -b & -c & -1/4 & -1/2 \\ -a & -a & -b & -b & -c & -1/4 & -1/2 \\ -b & -b & -b & -b & -c & -1/4 & 0 \\ -b & -b & -b & -b & -c & -1/4 & 0 \\ -c & -c & -c & -c & -c & 0 & 0 \\ -1/4 & -1/4 & -1/4 & -1/4 & 0 & 1/4 & 0 \\ -1/2 & -1/2 & 0 & 0 & 0 & 0 & 1/2 \end{pmatrix},$$

where $a > 0$ is a sufficiently large parameter, $b = a + 1/4$, and $c = a + 3/4$. For the block sequence (1.1), the alternative construction from §3 yields

$M =$

$$(1.8) \quad -\begin{pmatrix} a & b & b & b & c & d & d & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 1 & 1 \\ b & b & b & b & c & d & d & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 0 & 0 \\ b & b & b & b & c & d & d & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 0 & 0 \\ b & b & b & b & c & d & d & \frac{1}{5} & \frac{1}{5} & \frac{1}{4} & 0 & 0 \\ c & c & c & c & c & d & d & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\ d & d & d & d & d & d & d & 0 & 0 & 0 & 0 & 0 \\ d & d & d & d & d & d & d & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & -\frac{1}{5} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

where $a > 0$ is a sufficiently large parameter, $b = a + 2$, $c = a + 9/4$, and $d = a + 53/20$. That (1.6-1.8) are CdV matrices and how they were constructed will follow from the proof of Theorem 1 and §3.

The *Colin de Verdière graph parameter* $\mu(G)$ is defined to be the largest co-rank of a CdV matrix, that is the dimension of the largest nullspace among all CdV matrices associated to G. In [CdV], Colin de Verdière proved the following theorem.

Theorem 3. *The Colin de Verdière graph parameter $\mu(G) \leq 1$ if and only if G is a disjoint union of paths.*

The Colin de Verdière graph parameter $\mu(G) \leq 2$ if and only if G is outerplanar.

The Colin de Verdière graph parameter $\mu(G) \leq 3$ if and only if G is planar.

From [RST] and [LoSch] we have the additional characterization given by the following theorem.

Theorem 4. *The Colin de Verdière graph parameter $\mu(G) \leq 4$ if and only if G is linklessly embeddable in $\mathbb{R}^3$.*

See also [HLS] for a summary of these and other results.

The rank of the matrix (1.7) is 3, hence $\mu(G) \geq 4$ in this case. As G is linklessly embeddable in $\mathbb{R}^3$, we have $\mu(G) = 4$ for this graph. The rank of the matrix (1.8) is 6, hence $\mu(G) \geq 6$ for this graph.

2. Proof of Theorem 1

In this section we prove Theorem 1 and construct an optimal Colin de Verdière matrix for each threshold graph.

The following result is our main tool in the proof of Theorem 1.

Theorem 5 ([CdV],[HLS])**.** *Let v be a vertex of G. Then $\mu(G) \leq \mu(G-v) + 1$. If v is connected to all other vertices and $G-v$ is not $\overline{K_2}$ or empty, then equality holds.*

We also need a construction of the optimal matrix whose existence is asserted in the second part of Theorem 5. This has been given explicitly in [HLS] for the case when $G - v$ is connected. It is possible to extend this construction for an arbitrary $G - v$ but we shall only require the case where all connected components but one of $G - v$ are isolated vertices. Our proof naturally follows that of [HLS] with the requisite addition.

Lemma 2.1. *Let G be a graph on n vertices and let v be a vertex of G. Suppose that v is connected to all other vertices and that $G-v$ is not $\overline{K_2}$ or empty. Without loss of generality suppose that the first k vertices of $G-v$ induce a connected component C of $G-v$ and that the other $n - k - 1$ vertices of $G-v$ are isolated. (Possibly, $n - k - 1 = 0$).*

Now let $M^{'}$ be a Colin de Verdiére matrix for C with negative eigenvalue λ_1. Let z be a unit negative eigenvector of $M^{'}$ corresponding to λ_1. Also, without loss of generality we may assume that allf the square submatrices associated to the $n - k - 1$ isolates, $M^{'}_{k+1} = M^{'}_{k+2} = \ldots = M^{'}_{n-1} = 1$.

Now let $\theta = \sqrt{1 - (n - k - 1)\lambda_1}$ and let

$$M = \begin{pmatrix} M^{'} & 0_{k,n-k-1} & \theta z \\ 0_{n-k-1,k} & I_{n-k-1} & -\mathbf{1}_{n-k-1} \\ \theta z^T & -\mathbf{1}^T_{n-k-1} & \lambda_1^{-1} \end{pmatrix}.$$

Then M is a Colin de Verdiére matrix for G and $corank(M) = corank(M^{'}) + 1$.

Proof. M is obviously a discrete Schrödinger operator of G. Also it is easy to verify that M possesses the Strong Arnold property since $M^{'}$ does so.

Now let $x \in \ker M'$. The vectors of the form $(x, 0, 0)^T$ all belong to $\ker M$. It is not hard to verify that so does the vector $(\theta z, \lambda \mathbf{1}_{n-k-1}^T, \lambda)^T$. Therefore $corank(M) \geq corank(M') + 1$ and by the first part of Theorem 2.1 the equality of coranks follows.

The uniqueness of the negative eigenvalue of M follows via interlacing from the uniqueness of the negative eigenvalue of M' and $corank(M) = corank(M') + 1$. $\square$

We shall also need the following lemmas:

Lemma 2.2 ([HLS]). *If G has at least one edge then $\mu(G) = \max_K \mu(K)$, where K runs over all connected components of G.*

Lemma 2.3. *For $q \leq 2$, $\mu(K_{1,q}) = 1$ and for $q \geq 3$, $\mu(K_{1,q}) = 2$.*
An optimal Colin de Verdiére matrix for $\mu(K_{1,1})$ is

$$\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

For $2 \leq q \leq 3$, an optimal Colin de Verdiére matrix for $\mu(K_{1,q})$ is

$$M = \begin{pmatrix} 0 & -\mathbf{1} \\ -\mathbf{1}^T & \mathbf{0} \end{pmatrix}.$$

For $q \geq 4$, an optimal Colin de Verdiére matrix for $\mu(K_{1,q})$ is

$$M = \begin{pmatrix} 0 & -\mathbf{1} & -\mathbf{1} \\ -\mathbf{1}^T & I_{q-3} & \mathbf{0} \\ -\mathbf{1}^T & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Proof. The values of μ for the stars can be read off Theorem 3. As for the matrices, they can be seen to have the Strong Arnold Property by direct verification. $\square$

Proof of Theorem 1. Suppose that the second cone vertex in the construction sequence has been added at stage k (recall that by our convention the sequence always starts with a cone). Therefore before stage k the graph had been edgeless whereas after stage k it is in fact $K_{1,k-1}$. An optimal Colin de Verdiére matrix for it is given by Lemma 2.3.

We can trace the effect every further stage of the execution of the construction sequence has. When we add an isolate Lemma 2.2 implies that we do not alter the Colin de Verdiére number. If M is the matrix we had constructed so far, $M \oplus (1)$ is an optimal Colin de Verdiére matrix after the addition of an isolate.

When we add a cone Theorem 5 implies that we do increase the Colin de Verdiére number by 1. An optimal Colin de Verdiére matrix for the graph after the addition of a cone can be now obtained by the construction of Lemma 2.1.

It remains to observe that after stage k we have $c - 2$ cone additions left to do and thus the Colin de Verdiére number of G is $\mu(K_{1,k-1}) + c - 2$. $\square$

3. An alternative Colin de Verdière matrix

In this section we construct an alternative optimal Colin de Verdière matrix for all but case (3) in Theorem 1. There are three main steps, constructing a real-symmetric matrix M satisfying (1.3) and proving it has the appropriate co-rank, proving M satisfies (1.4), and proving M satisfies (1.5).

Construction of M.

We adopt the following labeling conventions. By $\mathbf{1}_{k\times m}$ we denote the $k \times m$ matrix of 1s, and by $\mathbf{0}_{k\times m}$ the $k \times m$ matrix of 0s. By $\mathbf{I}_{m\times m}$ we denote the $m \times m$ identity matrix.

Let G be a graph with degree and block sequence as in the statement of Theorem 1. We will construct symmetric, real valued, $n \times n$ matrices in blocks of rows and columns corresponding to the block sequence of construction for a connected threshold graph. For such a matrix, let $r_1, r_2, \ldots r_n$ denote the individual rows, and $R_1, R_2, \ldots, R_{2m+1}$ denote the row blocks. Here R_j has k_{m+2-j} rows for $1 \le j \le m+1$ corresponding to the blocks of cones and i_{j-m-1} rows for $m+2 \le j \le 2m+1$ corresponding to the blocks of isolates. Let $c_1 = r_1^T, c_2 = r_2^T, \ldots c_n = r_n^T$ denote the individual columns, and $C_1 = R_1^T, C_2 = R_2^T, \ldots, C_{2m+1} = R_{2m+1}^T$ denote the column blocks. For convenience, we will also write the expression $r_j + R_k$ to mean "row $j + \sum$(rows in block R_k)" whenever unambiguous.

We construct a family of CdV matrices for G. Let $\alpha_1, \ldots, \alpha_{2m+1} > 0$ be a set of parameters to be fixed later in the proof. For our construction of M, first take

$$R_1 \;\; = \;\; \Big(-\alpha_1 \mathbf{1}_{k_{m+1}\times k_{m+1}}, -\alpha_2 \mathbf{1}_{k_{m+1}\times k_m}, \ldots, -\alpha_{m+1}\mathbf{1}_{k_{m+1}\times k_1};$$

$$-\alpha_{m+2}\mathbf{1}_{k_{m+1}\times i_1}, \ldots, -\alpha_{2m+1}\mathbf{1}_{k_{m+1}\times i_m} \Big),$$

and

$$R_{2m+1} \;\; = \;\; \Big(-\alpha_{2m+1}\mathbf{1}_{i_m\times k_{m+1}}, \mathbf{0}_{i_m\times k_m}, \ldots, \mathbf{0}_{i_m\times k_1};$$

$$\mathbf{0}_{i_m\times i_1}, \ldots, \mathbf{0}_{i_m\times i_{m-1}}, \alpha_{2m+1}\mathbf{I}_{i_m\times i_m} \Big).$$

The idea here is that R_1 has only one independent row, and adding R_{2m+1} to r_1 will kill the last block in r_1, which can then be used to kill R_2. More precisely,

$$r_1 + R_{2m+1} \;\; = \;\; \Big((-\alpha_1 - i_m\alpha_{2m+1})\mathbf{1}_{1\times k_{m+1}}, -\alpha_2\mathbf{1}_{1\times k_m}, \ldots -\alpha_{m+1}\mathbf{1}_{1\times k_1};$$

$$-\alpha_{m+2}\mathbf{1}_{1\times i_1}, \ldots, -\alpha_{2m}\mathbf{1}_{1\times i_{m-1}}, \mathbf{0}_{1\times i_m} \Big),$$

which is equal to r_j for $k_{m+1} + 1 \le j \le k_{m+1} + k_m$ as long as

$$(3.1) \qquad\qquad\qquad \alpha_2 = \alpha_1 + i_m\alpha_{2m+1}.$$

Hence with this choice of α_2, R_2 is dependent. Similarly, for $2 \le j \le m$, we can arrange for each row of R_j to equal

$$r_1 + R_{2m+1} + \ldots + R_{2m+3-j},$$

provided for $2 \le j \le m+1$,

$$R_j \;\; = \;\; \Big(-\alpha_j \mathbf{1}_{k_{m+2-j}\times k_{m+1}}, -\alpha_j\mathbf{1}_{k_{m+2-j}\times k_m}, \ldots, -\alpha_j\mathbf{1}_{k_{m+2-j}\times k_{m+2-j}},$$

$$-\alpha_{j+1}\mathbf{1}_{k_{m+2-j}\times k_{m+3-j}}, \ldots, -\alpha_m\mathbf{1}_{k_{m+2-j}\times k_2}, -\alpha_{m+1}\mathbf{1}_{k_{m+2-j}\times k_1};$$

$$-\alpha_{m+2}\mathbf{1}_{k_{m+2-j}\times i_1}, \ldots, -\alpha_{2m+2-j}\mathbf{1}_{k_{m+2-j}\times i_{m+1-j}},$$

$$\mathbf{0}_{k_{m+2-j}\times i_{m+2-j}}, \ldots, \mathbf{0}_{k_{m+2-j}\times i_m} \Big),$$

for $m + 2 \leq j \leq 2m + 1$

$$
\begin{aligned}
R_j \quad = \quad & \Big(-\alpha_j \mathbf{1}_{i_{j-m-1} \times k_{m+1}}, -\alpha_j \mathbf{1}_{i_{j-m-1} \times k_m}, \ldots, -\alpha_j \mathbf{1}_{i_{j-m-1} \times k_{j-m}}, \\
& \mathbf{0}_{i_{j-m-1} \times k_{j-m+1}}, \ldots, \mathbf{0}_{i_{j-m-1} \times k_1}; \\
& \mathbf{0}_{i_{j-m-1} \times i_1}, \ldots, \mathbf{0}_{i_{j-m-1} \times i_{j-m-2}}, \alpha_j \mathbf{I}_{i_{j-m-1} \times i_{j-m-1}}, \\
& \mathbf{0}_{i_{j-m-1} \times i_{j-m}}, \ldots, \mathbf{0}_{i_{j-m-1} \times i_m} \Big)
\end{aligned}
$$

and for $2 \leq j \leq m + 1$,

$$
(3.2) \qquad \alpha_j = \alpha_1 + i_m \alpha_{2m+1} + i_{m-1} \alpha_{2m} + \ldots + i_{m+2-j} \alpha_{2m+3-j}.
$$

That is, $\alpha_2, \ldots, \alpha_{m+1}$ depend on $\alpha_1, \alpha_{m+2}, \ldots, \alpha_{2m+1}$ and the space of available parameters has dimension $m + 1$.

We have shown

$$
\operatorname{rank} M = 1 + \sum_{j=1}^{m} i_j,
$$

and M satisfies (1.3).

Proof of property (1.4)**.**

For (1.4) we will prove a result on the structure of the characteristic polynomial. We will employ multi-index notation: Let $\epsilon \in \mathbb{Z}^l$,

$$
\epsilon = (\epsilon_1, \ldots, \epsilon_l).
$$

For a vector $x \in \mathbb{R}^l$, by x^ϵ, we mean

$$
x^\epsilon = x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_l^{\epsilon_l}.
$$

By $|\epsilon|$, we mean

$$
|\epsilon| = \epsilon_1 + \ldots + \epsilon_l,
$$

and for $\epsilon, \epsilon' \in \mathbb{Z}^l$, by $\epsilon \leq \epsilon'$, we mean

$$
\epsilon_j \leq \epsilon'_j
$$

for each $1 \leq j \leq l$, and define also $\epsilon - \epsilon'$ and $\epsilon + \epsilon'$ componentwise.

We also use the following labeling convention when unambiguous: Let

$$
\begin{aligned}
i \quad & = \quad \sum_{j=1}^{m} i_j, \\
k \quad & = \quad \sum_{j=1}^{m+1} k_j, \text{ and} \\
\alpha \quad & = \quad \alpha_{m+1} = \alpha_1 + \sum_{j=1}^{m} i_j \alpha_{m+1+j},
\end{aligned}
$$

and define corresponding vectors

$$
\begin{aligned}
\bar{i} \quad & = \quad (i_1, \ldots, i_m), \text{ and} \\
\bar{\alpha} \quad & = \quad (\alpha_{m+2}, \ldots, \alpha_{2m+1}).
\end{aligned}
$$

Observe $\bar{\alpha}$ does not have an entry for α_1, and has the same length as $\bar{i}$. We define also the vector

$$\bar{\lambda} \;\; = \;\; \underbrace{(\lambda, \ldots, \lambda)}_{m},$$

so that for a multi-index $\epsilon \in \mathbb{Z}^m$, we have

$$(\bar{\alpha} - \bar{\lambda})^\epsilon = (\alpha_{m+2} - \lambda)^{\epsilon_1} \cdots (\alpha_{2m+1} - \lambda)^{\epsilon_m}.$$

To aid in computation, we introduce another set of parameters. For $1 \leq j \leq m$, define

$$\beta_j = k_{m+1} + k_m + \ldots + k_{j+1},$$

so that $\beta_j < \beta_{j-1}$ and

$$\beta_{j-p} - \beta_j = k_j + \ldots + k_{j-p+1}.$$

We choose

(3.3)
$$\alpha_{m+1+j} = \frac{1}{\beta_j},$$

leaving α_1 free.

Let M be the matrix constructed above. We will calculate $\det(M - \lambda \mathbf{I})$ by first using a similarity transformation to produce rows of zeros in M, and then using properties of the det function.

Proposition 3.1. *Suppose M is a real symmetric matrix constructed according to the algoritheorem above with this choice of the α_j. Then if $\alpha_1 > 0$ is chosen sufficiently large, there exist positive constants γ and c_ϵ, for each $\epsilon \in \{0,1\}^m$, $|\epsilon| \geq 0$, such that*

$$\det(M - \lambda \mathbf{I}) = (-\lambda)^{k-1} \left((-\gamma - \lambda)(\bar{\alpha} - \bar{\lambda})^{\bar{i}} - \sum_{|\epsilon| \geq 0} c_\epsilon (\bar{\alpha} - \bar{\lambda})^{\bar{i} - \epsilon} \right).$$

Remark. The benefit of Proposition 3.1 is that we can immediately conclude that the spectrum of M contains $k - 1$ zeros, and if $\lambda < 0$, $(1/\beta_j - \lambda) > 0$ implies

$$(-\lambda)^{1-k} \det(M - \lambda \mathbf{I}) = 0$$

can be rearranged into an equation of the form

$$f(\lambda) = g(\lambda),$$

with $f(\lambda) = \lambda$ and

$$g(\lambda) = -\sum_{|\epsilon| \geq 0} c_\epsilon (\bar{\alpha} - \bar{\lambda})^{-\epsilon} - \gamma.$$

Now $f'(\lambda) = 1$ and $g'(\lambda) < 0$ for $\lambda < 0$ implies f and g can intersect at most at one point for $\lambda < 0$. But since the trace of M is negative by construction, we conclude there is precisely one negative eigenvalue.

Proof of Proposition 3.1. The choices of $\alpha_2, \ldots, \alpha_{m+1}$ depending on the other α_j were made precisely so that through row operations we can reduce M to a matrix

$$PM = \begin{pmatrix} -\alpha \mathbf{1}_{1 \times k} & \mathbf{0}_{1 \times i} \\ \mathbf{0}_{(k-1) \times k} & \mathbf{0}_{(k-1) \times i} \\ M'_{2,1} & M'_{2,2} \end{pmatrix},$$

where $(M'_{2,1}, M'_{2,2})$ is the unchanged last i rows from M. Here P is the invertible matrix whose action by left multiplication is these row operations. Computing the action of P^{-1} by right multiplication produces the corresponding similarity transformation, and M is similar to a matrix

$$PMP^{-1} = \begin{pmatrix} -k\alpha & -\alpha\mathbf{1}_{1\times(k-1)} & r_{1,3} \\ \mathbf{0}_{(k-1)\times 1} & \mathbf{0}_{(k-1)\times(k-1)} & \mathbf{0}_{(k-1)\times i} \\ c_{1,3} & M''_{3,2} & \tilde{B} \end{pmatrix},$$

where

$$r_{1,3} = (\beta_1\alpha\mathbf{1}_{1\times i_1}, \beta_2\alpha\mathbf{1}_{1\times i_2}, \ldots, \beta_m\alpha\mathbf{1}_{1\times i_m}),$$
$$(3.4) \qquad c_{1,3} = -\mathbf{1}_{i\times 1},$$

and $M''_{3,2}$ is the $i \times (k-1)$ sub-matrix remaining unchanged from M. Here, (3.4) follows from (3.3) and $\tilde{B}$ is the $i \times i$ matrix given in row blocks:

$$(3.5) \qquad \tilde{B} = \begin{pmatrix} \widetilde{R}_1 \\ \widetilde{R}_2 \\ \vdots \\ \widetilde{R}_m \end{pmatrix},$$

with

$$\widetilde{R}_1 = \left(\mathbf{1}_{i_1\times i_1} + \beta_1^{-1}\mathbf{I}_{i_1\times i_1}, \frac{\beta_2}{\beta_1}\mathbf{1}_{i_1\times i_2}, \frac{\beta_3}{\beta_1}\mathbf{1}_{i_1\times i_3}, \ldots, \frac{\beta_m}{\beta_1}\mathbf{1}_{i_1\times i_m} \right)$$

$$\widetilde{R}_2 = \left(\mathbf{1}_{i_2\times i_1}, \mathbf{1}_{i_2\times i_2} + \beta_2^{-1}\mathbf{I}_{i_2\times i_2}, \frac{\beta_3}{\beta_2}\mathbf{1}_{i_2\times i_3}, \ldots, \frac{\beta_m}{\beta_2}\mathbf{1}_{i_1\times i_m} \right),$$

$$\widetilde{R}_3 = \left(\mathbf{1}_{i_3\times i_1}, \mathbf{1}_{i_3\times i_2}, \mathbf{1}_{i_3\times i_3} + \beta_3^{-1}\mathbf{I}_{i_3\times i_3}, \frac{\beta_4}{\beta_3}\mathbf{1}_{i_3\times i_4}, \ldots, \frac{\beta_m}{\beta_3}\mathbf{1}_{i_3\times i_m} \right),$$

$$\vdots$$

$$\widetilde{R}_{m-1} = \left(\mathbf{1}_{i_{m-1}\times i_1}, \ldots, \mathbf{1}_{i_{m-1}\times i_{m-2}}, \right.$$

$$\left. \mathbf{1}_{i_{m-1}\times i_{m-1}} + \beta_{m-1}^{-1}\mathbf{I}_{i_{m-1}\times i_{m-1}}, \frac{\beta_m}{\beta_{m-1}}\mathbf{1}_{i_{m-1}\times i_m} \right),$$

$$\widetilde{R}_m = \left(\mathbf{1}_{i_m\times i_1}, \ldots, \mathbf{1}_{i_m\times i_{m-1}}, \mathbf{1}_{i_m\times i_m} + \beta_m^{-1}\mathbf{I}_{i_m\times i_m} \right).$$

Now PMP^{-1} has rows of zero for r_2 through r_k. Since similarity transformations leave the spectrum invariant, we have

$$\det(M - \lambda\mathbf{I}) = \det(PMP^{-1} - \lambda\mathbf{I})$$
$$= (-\lambda)^{k-1}\det(M_0),$$

where M_0 is the $(i+1) \times (i+1)$ matrix

$$M_0 = \begin{pmatrix} -k\alpha - \lambda & r_{1,3} \\ c_{1,3} & \tilde{B} - \lambda\mathbf{I}_{i\times i} \end{pmatrix}.$$

The following lemma is the induction step of the proof. In order to simplify notation, set

$$(3.6) \qquad B = \tilde{B} - \mathbf{1}_{i\times i},$$

and for tR_j defined above,

$$(3.7) \qquad\qquad\qquad R_j = \widetilde{R}_j - \mathbf{1}_{i_j \times i}.$$

Lemma 3.2. *For each $1 \leq j \leq m-1$, let $I_j = i_m + i_{m-1} + \ldots + i_{m-j+1}$ and $\bar{I}_j = (i_m, i_{m-1}, \ldots, i_{m-j+1})$. There exist positive constants c_ϵ and $(i - I_j + 1) \times (i - I_j + 1)$ matrices M_ϵ^j for each $\epsilon \in \{0,1\}^j$, $c_0 = 1$, such that*

$$(3.8) \qquad\qquad \det(M_0) = \sum_{\epsilon \in \{0,1\}^j} c_\epsilon (\bar{\alpha} - \bar{\lambda})^{(\bar{I}_j - \epsilon)} \det(M_\epsilon^j).$$

The M_ϵ^j satisfy the following properties:

(i)

$$(3.9) \qquad\qquad M_0^j = \begin{pmatrix} -\gamma_j - \lambda & r_j \\ -\mathbf{1}_{(i-I_j) \times 1} & B^j - \lambda \mathbf{I}_{(i-I_j) \times (i-I_j)} \end{pmatrix},$$

where

$$(3.10) \qquad\qquad\qquad \gamma_j = k\alpha - I_j > 0,$$

$$\begin{aligned} r_j &= \big((\alpha(\beta_1 - k) - \lambda)\mathbf{1}_{1 \times i_1}, (\alpha(\beta_2 - k) - \lambda)\mathbf{1}_{1 \times i_2}, \\ &\qquad \ldots, (\alpha(\beta_{m-j} - k) - \lambda)\mathbf{1}_{1 \times i_{m-j}}\big), \end{aligned}$$

and B^j is the $(i - I_j) \times (i - I_j)$ matrix obtained from B in (3.6) by removing the last I_j rows and columns.

(ii) For $\epsilon \neq 0$,

$$(3.11) \qquad\qquad M_\epsilon^j = \begin{pmatrix} -\gamma_\epsilon^j & r_j \\ c_\epsilon^j & B^j - \lambda \mathbf{I}_{(i-I_j) \times (i-I_j)} \end{pmatrix},$$

where r_j and B_j are as in (i), $\gamma_\epsilon^j > 0$, and

$$c_\epsilon^j = \begin{pmatrix} -\delta_\epsilon^1 \mathbf{1}_{i_1 \times 1} \\ -\delta_\epsilon^2 \mathbf{1}_{i_2 \times 1} \\ \vdots \\ -\delta_\epsilon^{m-j} \mathbf{1}_{i_{m-j} \times 1} \end{pmatrix},$$

for constants

$$\delta_\epsilon^p > 0.$$

(iii) We have the relations

$$\begin{aligned} \delta_{(\epsilon,1)}^p &= \left(1 - \frac{\beta_{m-j}}{\beta_p}\right), \\ \delta_{(\epsilon,0)}^p &= \delta_\epsilon^p, \end{aligned}$$

for $1 \leq p \leq m-j-1$, and

$$\begin{aligned} -\gamma_{(\epsilon,1)}^{j+1} &= \alpha(\beta_{m-j} - k) - \beta_{m-j}^{-1}, \\ -\gamma_{(\epsilon,0)}^{j+1} &= -\gamma_\epsilon^j + i_{m-j} \delta_\epsilon^{m-j}. \end{aligned}$$

Proof. The basic idea is to inductively use row and column operations on the last remaining row block. At each step this results in the last row block being diagonal except for negative elements in the first column. When we expand the determinant along the last row block, we get terms involving $(\beta_j^{-1} - \lambda)^{i_j}$ and terms involving

$(\beta_j^{-1} - \lambda)^{i_j - 1}$. At each step in the induction, we then permute the last column to the first, keeping track of the powers of -1, to get a leading element of the form $-\gamma_j''$. We will carefully define all of this in the remainder of the proof.

Base case. The last row block of M_0 is of the form

$$\left(-\mathbf{1}_{i_m \times 1}, R_m^0\right),$$

where

$$R_m^0 = \left(\mathbf{1}_{i_m \times i_1}, \ldots, \mathbf{1}_{i_m \times i_{m-1}}, \mathbf{1}_{i_m \times i_m} + \beta_m^{-1} \mathbf{I}_{i_m \times i_m}\right).$$

To simplify this expression, we add column 1 to the last i columns, which has the effect of replacing $\tilde{B}$ with B as defined in (3.6), but adds $-k\alpha - \lambda$ to the last i elements in the first row. To eliminate the $-\lambda$ in the last i_m elements in the first row, we subtract the last i_m rows from the first and obtain

$$\det(M_0) = \det(M_0'),$$

where

$$M_0' = \begin{pmatrix} -\gamma_1 - \lambda & r_1' \\ -\mathbf{1}_{i \times 1} & B \end{pmatrix}.$$

Here

$$-\gamma_1 = -k\alpha + i_m,$$

$$\begin{aligned} r_1' &= \left((\alpha(\beta_1 - k) - \lambda)\mathbf{1}_{1 \times i_1}, (\alpha(\beta_2 - k) - \lambda)\mathbf{1}_{1 \times i_2},\right. \\ &\quad \left. \ldots, (\alpha(\beta_{m-1} - k) - \lambda)\mathbf{1}_{1 \times i_{m-1}}, -\gamma_1'\mathbf{1}_{1 \times i_m}\right), \end{aligned}$$

with

$$-\gamma_1' = \alpha(\beta_m - k) - \beta_m^{-1},$$

and B is as defined in (3.6).

Now when we expand the determinant of M_0' along the last row block, we get contributions from the -1s in the first column, and the diagonal elements in R_m. Expanding the determinant along these rows and permuting the resulting submatrices so that the lower right $(i - i_m) \times (i - i_m)$ submatrix agrees with the definition of B^1 in the lemma, we obtain

$$\det(M_0') = (\beta_m^{-1} - \lambda)^{i_m} \det(M_1) + i_m(\beta_m^{-1} - \lambda)^{i_m - 1} \det(M_1'),$$

where

$$M_1 = \begin{pmatrix} -\gamma_1 - \lambda & r_1 \\ -\mathbf{1}_{(i - i_m) \times 1} & B^1 \end{pmatrix}$$

and

$$M_1' = \begin{pmatrix} -\gamma_1' & r_1 \\ c_1 & B^1 \end{pmatrix}.$$

Here B^1 is defined in the statement of the lemma,

$$r_1 = \left(\beta_1 \alpha, \beta_2 \alpha, \ldots, \beta_{m-1} \alpha\right),$$

and

$$c_1 = \begin{pmatrix} -\delta_1^1 \mathbf{1}_{i_1 \times 1} \\ -\delta_2^1 \mathbf{1}_{i_2 \times 1} \\ \vdots \\ -\delta_{m-1}^1 \mathbf{1}_{i_{m-1} \times 1} \end{pmatrix},$$

with

$$\delta_p^1 = 1 - \frac{\beta_m}{\beta_p},$$

in accordance with the statement of the Lemma.

Induction step. Now suppose the Lemma is true for some $1 \le j \le m - 2$. We show the same reduction ideas used in the base case will reduce to the statement of the Lemma for $j + 1$. That is, assume we have matrices and constants as in the formula (3.8). For M_0^j, we subtract the last i_{m-j} rows from the first to eliminate the $-\lambda$ in the last i_{m-j} elements in the first row. Expanding the determinant along the last row block (the last i_{m-j} rows) and permuting as necessary yields

$$\det(M_0^j) = (\beta_{m-j}^{-1} - \lambda)^{i_{m-j}} \det(M_0^{j+1}) + i_{m-j}(\beta_{m-j}^{-1} - \lambda)^{i_{m-j}-1} \det(M_1^{j+1}),$$

where M_0^{j+1} is defined in the statement of the Lemma, and

$$M_1^{j+1} = \begin{pmatrix} -\gamma_{j+1}^1 & r_{j+1} \\ c_{j+1}^1 & B^{j+1} \end{pmatrix},$$

where

$$-\gamma_{j+1}^1 = \alpha(\beta_{m-j} - k) - \beta_{m-j}^{-1},$$

r_{j+1} is as defined in the Lemma, and

$$c_{j+1}^1 = \begin{pmatrix} -\delta_1^{j+1,1} \mathbf{1}_{i_1 \times 1} \\ -\delta_2^{j+1,1} \mathbf{1}_{i_2 \times 1} \\ \vdots \\ -\delta_{m-j-1}^{j+1,1} \mathbf{1}_{i_{m-j-1} \times 1} \end{pmatrix},$$

with

(3.12) $$-\delta_p^{j+1,1} = \frac{\beta_{m-j}}{\beta_p} - 1 < 0.$$

Observe the multi-index ϵ_1 associated to M_1^{j+1} is

$$\epsilon_1 = (\underbrace{0, \ldots, 0}_{j}, 1),$$

so $\delta_{\epsilon_1}^p := \delta_p^{j+1,1}$ in (3.12) agrees with the statement of the Lemma.

We next tackle M_ϵ^j for $|\epsilon| > 0$. We again subtract the last i_{m-j} rows from the first row to eliminate the $-\lambda$s in the last i_{m-j} columns. As with the other cases, we then expand the determinant along the last row block and permute as necessary to ensure B^{j+1} be the lower right submatrix. We have

$$\det M_\epsilon^j =$$
$$= (\beta_{m-j}^{-1} - \lambda)^{i_{m-j}} \det(M_{\epsilon_2}^{j+1}) + i_{m-j}\delta_\epsilon^{m-j}(\beta_{m-j}^{-1} - \lambda)^{i_{m-j}-1} \det(M_{\epsilon_3}^{j+1}),$$

where

$$M_{\epsilon_2}^{j+1} = \begin{pmatrix} -\gamma_\epsilon^j & r_{j+1} \\ c_{\epsilon_2}^{j+1} & B^{j+1} \end{pmatrix},$$

with γ_ϵ^j inherited from M_ϵ^j, r_{j+1} as defined in the Lemma, and

$$c_{\epsilon_2}^{j+1} = \begin{pmatrix} -\delta_\epsilon^1 \mathbf{1}_{i_1 \times 1} \\ -\delta_\epsilon^2 \mathbf{1}_{i_2 \times 1} \\ \vdots \\ -\delta_\epsilon^{m-j-1} \mathbf{1}_{i_{m-j-1} \times 1} \end{pmatrix},$$

with δ_ϵ^p inherited from M_ϵ^j. Here

$$M_{\epsilon_3}^{j+1} = \begin{pmatrix} -\gamma_{\epsilon_3}^{j+1} & r_{j+1} \\ c_{\epsilon_3}^{j+1} & B^{j+1} \end{pmatrix},$$

where

$$-\gamma_{\epsilon_3}^{j+1} = \alpha(\beta_{m-j} - k) - \beta_{m-j}^{-1},$$

and

$$c_{\epsilon_3}^{j+1} = \begin{pmatrix} -\delta_{\epsilon_3}^{j+1,1} \mathbf{1}_{i_1 \times 1} \\ -\delta_{\epsilon_3}^{j+1,2} \mathbf{1}_{i_2 \times 1} \\ \vdots \\ -\delta_{\epsilon_3}^{j+1,m-j-1} \mathbf{1}_{i_{m-j-1} \times 1} \end{pmatrix},$$

with

$$-\delta_{\epsilon_3}^{j+1,p} = \frac{\beta_{m-j}}{\beta_p} - 1,$$

as in the statement of the Lemma.

Relabeling as necessary, this completes the proof of the induction step, and hence the proof of the Lemma. □

Now in order to finish the proof of Proposition 3.1 we calculate what happens in the $j = m$ case. From Lemma 3.2, we have

$$\det(M_0) = \sum_{\epsilon \in \{0,1\}^{m-1}} c_\epsilon (\bar{\alpha} - \bar{\lambda})^{(\bar{I}_{m-1} - \epsilon)} \det(M_\epsilon^{m-1}),$$

where, relabeling for simplicity in exposition,

$$M_0^{m-1} = \begin{pmatrix} -\gamma - \lambda & (\alpha(\beta_1 - k) - \lambda)\mathbf{1}_{1 \times i_1} \\ -\mathbf{1}_{i_1 \times 1} & (\beta_1^{-1} - \lambda)\mathbf{I}_{i_1 \times i_1} \end{pmatrix},$$

with $-\gamma = -k\alpha + i - i_1$, and

$$M_\epsilon^{m-1} = \begin{pmatrix} -\gamma_\epsilon & (\alpha(\beta_1 - k) - \lambda)\mathbf{1}_{1 \times i_1} \\ -\delta_\epsilon \mathbf{1}_{i_1 \times 1} & (\beta_1^{-1} - \lambda)\mathbf{I}_{i_1 \times i_1} \end{pmatrix}.$$

Using *iii* from the Lemma, if $\epsilon = (\epsilon', 1)$ with $\epsilon' \in \{0,1\}^{m-2}$,

$$-\gamma_{(\epsilon',1)} = \alpha(\beta_2 - k) - \beta_2^{-1},$$

and if $\epsilon = (\epsilon', 0)$,

$$
\begin{aligned}
-\gamma_{(\epsilon',0)} &= -\gamma^{m-1}_{(\epsilon',0)} \\
&= -\gamma^2_{\epsilon'} + i_2\delta^2_{\epsilon'} \\
&< -\gamma^2_{\epsilon'} + i_2 \\
&\leq -\gamma^j,
\end{aligned}
$$

by induction.

Proceeding as in the proof of the Lemma yields

$$
\det(M_0^{m-1}) = (-\gamma' - \lambda)(\beta_1^{-1} - \lambda)^{i_1} - i_1\gamma''(\beta_1^{-1} - \lambda)^{i_1-1},
$$

where

$$
\begin{aligned}
-\gamma' &= -k\alpha + i \text{ and} \\
-\gamma'' &= \alpha(\beta_1 - k) - \beta_1^{-1}
\end{aligned}
$$

both of which are negative. Similarly,

$$
\det(M_\epsilon^{m-1}) = -\gamma_\epsilon(\beta_1^{-1} - \lambda)^{i_1} - i_m\delta_\epsilon\gamma'_\epsilon(\beta_1^{-1} - \lambda)^{i_1-1},
$$

where γ'_ϵ was defined above, and is negative. This proves the Proposition.

$\square$

Remark. We illustrate the proof of Proposition 3.1 by following the steps in the concrete example of $M = (1.8)$. We have

$$
\det(M - \lambda\mathbf{I}) = \det(PMP^{-1} - \lambda\mathbf{I}),
$$

with

$$
PMP^{-1} =
$$

$$
\begin{pmatrix}
-7\alpha & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha & -\alpha & 5\alpha & 5\alpha & 4\alpha & \alpha & \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & \frac{6}{5} & 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} \\
-1 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & 0 & 0 & 1 & \frac{6}{5} & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} \\
-1 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 1 & 1 & \frac{5}{4} & \frac{1}{4} & \frac{1}{4} \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2
\end{pmatrix},
$$

where $\alpha = \alpha_1 + 53/20$ for $\alpha_1 > 0$ yet to be determined. The rows of zeros yield a contribution to $\det(M - \lambda\mathbf{I})$ of $(-\lambda)^6$, and we have reduced to studying $\det(M_0)$, for

$$
M_0 = \begin{pmatrix}
-7\alpha - \lambda & 5\alpha & 5\alpha & 4\alpha & \alpha & \alpha \\
-1 & \frac{6}{5} - \lambda & 1 & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} \\
-1 & 1 & \frac{6}{5} - \lambda & \frac{4}{5} & \frac{1}{5} & \frac{1}{5} \\
-1 & 1 & 1 & \frac{5}{4} - \lambda & \frac{1}{4} & \frac{1}{4} \\
-1 & 1 & 1 & 1 & 2 - \lambda & 1 \\
-1 & 1 & 1 & 1 & 1 & 2 - \lambda
\end{pmatrix}.
$$

We use the first column to kill the 1s in the last three rows, and then the last two rows to kill the $-\lambda$s in the first row to get $\det(M_0) = \det(M_0')$, where

$$M_0' =$$

$$\begin{pmatrix} -7\alpha+2-\lambda & -2\alpha-\lambda & -2\alpha-\lambda & -3\alpha-\lambda & -6\alpha-1 & -6\alpha-1 \\ -1 & \frac{1}{5}-\lambda & 0 & -\frac{1}{5} & -\frac{4}{5} & -\frac{4}{5} \\ -1 & 0 & \frac{1}{5}-\lambda & -\frac{1}{5} & -\frac{4}{5} & -\frac{4}{5} \\ -1 & 0 & 0 & \frac{1}{4}-\lambda & -\frac{3}{4} & -\frac{3}{4} \\ -1 & 0 & 0 & 0 & 1-\lambda & 0 \\ -1 & 0 & 0 & 0 & 0 & 1-\lambda \end{pmatrix}.$$

Expanding $\det(M_0')$ along the last two lines yields:

$$\det(M_0') = 2(1-\lambda)\det(M_1) + (1-\lambda)^2 \det(M_2),$$

where

$$M_1 = \begin{pmatrix} -6\alpha-1 & -2\alpha-\lambda & -2\alpha-\lambda & -3\alpha-\lambda \\ -\frac{4}{5} & \frac{1}{5}-\lambda & 0 & -\frac{1}{5} \\ -\frac{4}{5} & 0 & \frac{1}{5}-\lambda & -\frac{1}{5} \\ -\frac{3}{4} & 0 & 0 & \frac{1}{4}-\lambda \end{pmatrix},$$

and

$$M_2 = \begin{pmatrix} -7\alpha+2-\lambda & -2\alpha-\lambda & -2\alpha-\lambda & -3\alpha-\lambda \\ -1 & \frac{1}{5}-\lambda & 0 & -\frac{1}{5} \\ -1 & 0 & \frac{1}{5}-\lambda & -\frac{1}{5} \\ -1 & 0 & 0 & \frac{1}{4}-\lambda \end{pmatrix}.$$

Subtract the last row from the first in M_1 and M_2 and expanding the respective determinants along the last row yields

$$\det(M_1) = \frac{3}{4}\det(M_3) + \left(\frac{1}{4}-\lambda\right)\det(M_4),$$

$$\det(M_2) = \det(M_5) + \left(\frac{1}{4}-\lambda\right)\det(M_6),$$

where

$$M_3 = \begin{pmatrix} -3\alpha-\frac{1}{4} & -2\alpha-\lambda & -2\alpha-\lambda \\ -\frac{1}{5} & \frac{1}{5}-\lambda & 0 \\ -\frac{1}{5} & 0 & \frac{1}{5}-\lambda \end{pmatrix},$$

$$M_4 = \begin{pmatrix} -6\alpha-1 & -2\alpha-\lambda & -2\alpha-\lambda \\ -\frac{4}{5} & \frac{1}{5}-\lambda & 0 \\ -\frac{4}{5} & 0 & \frac{1}{5}-\lambda \end{pmatrix},$$

$$M_5 = \begin{pmatrix} -3\alpha-\frac{1}{4} & -2\alpha-\lambda & -2\alpha-\lambda \\ -\frac{1}{5} & \frac{1}{5}-\lambda & 0 \\ -\frac{1}{5} & 0 & \frac{1}{5}-\lambda \end{pmatrix}, \text{ and}$$

$$M_6 = \begin{pmatrix} -7\alpha+3-\lambda & -2\alpha-\lambda & -2\alpha-\lambda \\ -1 & \frac{1}{5}-\lambda & 0 \\ -1 & 0 & \frac{1}{5}-\lambda \end{pmatrix}.$$

Expanding the determinant as in the previous iterations yields:

$$\det(M_3) = \frac{2}{5}(-2\alpha - \frac{1}{5})\left(\frac{1}{5} - \lambda\right) + \left(-3\alpha + \frac{3}{20}\right)\left(\frac{1}{5} - \lambda\right)^2,$$

$$\det(M_4) = \frac{8}{5}\left(-2\alpha - \frac{1}{5}\right)\left(\frac{1}{5} - \lambda\right) + \left(-6\alpha + \frac{27}{20}\right)\left(\frac{1}{5} - \lambda\right)^2,$$

$$\det(M_5) = \det(M_3), \text{ and}$$

$$\det(M_6) = 2\left(-2\alpha - \frac{1}{5}\right)\left(\frac{1}{5} - \lambda\right) + (-7\alpha + 5 - \lambda)\left(\frac{1}{5} - \lambda\right)^2.$$

Following our calculations back to the original matrix M, we see $\det(M)$ satisfies Proposition 3.1 as long as $\alpha_1 > 0$ is chosen large enough so that $\alpha = \alpha_1 + 53/20$ satisfies the inequalities

$$-7\alpha + 5 < 0,$$
$$-3\alpha + \frac{3}{20} < 0, \text{ and}$$
$$-6\alpha + \frac{27}{20} < 0.$$

$\square$

Proof of property (1.5).

In order to verify (1.5), we write such a matrix X as

$$X = \left(\widetilde{C}_1, \ldots, \widetilde{C}_{2m+1}\right)$$

with $\widetilde{C}_j$ having the same dimensions as C_j from M. We also let $\widetilde{R}_j = \widetilde{C}_j^T$ be the row blocks of X. The assumptions on X and the construction of M show

$$\widetilde{C}_1 = \mathbf{0}_{n \times k_{m+1}}$$

and

$$\widetilde{C}_2 = \left(\begin{array}{c} \mathbf{0}_{(n-i_m) \times k_m} \\ \widetilde{C}^2_{i_m \times k_m} \end{array}\right),$$

where $\widetilde{C}^2_{i_m \times k_m}$ denotes an arbitary $i_m \times k_m$ matrix particular to $\widetilde{C}_2$. The equation $MX = 0$ implies

$$R_{2m+1}\widetilde{C}_2 = \mathbf{0}_{i_m \times k_m},$$

which further implies

$$\mathbf{I}_{i_m \times i_m}\widetilde{C}^2_{i_m \times k_m} = \mathbf{0}_{i_m \times k_m},$$

so $\widetilde{C}_2$ is zero, as well as $\widetilde{R}_2 = \widetilde{C}_2^T$. Continuing like this, for $3 \leq j \leq m+1$

$$R_{2m+1}\widetilde{C}_j = \mathbf{0}_{i_m \times k_{m+2-j}},$$

and for $m+2 \leq j \leq 2m+1$,

$$R_{2m+1}\widetilde{C}_j = \mathbf{0}_{i_m \times i_{j-m-1}},$$

together imply $\widetilde{R}_{2m+1} = \mathbf{0}_{i_m \times n}$. Now for the purposes of induction, suppose we know $\widetilde{R}_{2m+2-j} = \mathbf{0}_{i_{m+2-j} \times n}$, $\widetilde{C}_{j+1} = \mathbf{0}_{n \times k_{m+1-j}}$ for some $2 \leq j \leq m-1$. Multiplying $\widetilde{C}_{j+1+k}$ on the left by R_{2m+1-j} and setting equal to zero for $1 \leq k \leq m-j$

gives $\widetilde{R}_{2m+1-j} = \mathbf{0}_{i_{m-j} \times n}$ and hence $\widetilde{C}_{j+2} = \mathbf{0}_{n \times k_{m-j}}$. Thus by induction $X = 0$ and M satisfies (1.5).

Remark. In order to illustrate the proof of property (1.5) for M, we show how it works for our example (1.8). For this M, X has the form

$$X = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 & x_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_3 & x_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_5 & x_6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_7 & x_8 & x_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{15} & x_{16} & x_{17} & x_{18} & x_{19} \\
0 & 0 & 0 & 0 & 0 & x_{10} & x_{15} & 0 & x_{20} & x_{21} & x_{22} & x_{23} \\
0 & 0 & 0 & 0 & 0 & x_{11} & x_{16} & x_{20} & 0 & x_{24} & x_{25} & x_{26} \\
0 & 0 & 0 & 0 & x_7 & x_{12} & x_{17} & x_{21} & x_{24} & 0 & x_{27} & x_{28} \\
0 & x_1 & x_3 & x_5 & x_8 & x_{13} & x_{18} & x_{22} & x_{25} & x_{27} & 0 & x_{29} \\
0 & x_2 & x_4 & x_6 & x_9 & x_{14} & x_{19} & x_{23} & x_{26} & x_{28} & x_{29} & 0
\end{pmatrix}.$$

Multiplying X on the left by the last two rows of M implies the last two rows of X are zero. Hence the last two columns of X are zero, which from the structure of X implies the first 4 rows of X are zero. Now the 10th row of M has four non-zero entries followed zeros and a non-zero entry in the tenth position. This implies row 10 of X is zero, so column 10 is zero, and hence row 5 is zero. Continuing in this fashion eventually gives $X = 0$.

$\square$

Following the numbering schemes used in the proof of Proposition 3.1 we get the following estimates on the size of the eigenvalues of M.

Corollary 3.3. *If $\lambda < 0$ is the negative eigenvalue of M, then*

$$\lambda < -k\alpha + i.$$

If $\lambda > 0$ is a positive eigenvalue of M, then

$$\lambda \geq \beta_1^{-1} > \frac{1}{k}.$$

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