SEMICLASSICAL CONTROL AND $L^2$ RESTRICTION BOUNDS
FOR NEUMANN DATA ALONG HYPERSURFACES

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Abstract. We prove an $h$-microlocal control estimate on $h\delta$-scales ($0 \leq \delta < 1$) for eigenfunctions in a microlocal neighbourhood of a hypersurface $H$ in terms of the mass in the restricted coball bundle and the mass concentrated on the glancing set. We use these control estimates to obtain an $O(1)$ $L^2$-restriction bound for the Neumann data along $H$.

1. Introduction

We consider here the eigenvalue problem on a compact Riemannian manifold $(M, g)$ with or without boundary, with either Dirichlet or Neumann boundary conditions if $\partial M \neq \emptyset$. That is, we consider

$$
\begin{align*}
-\Delta_g \varphi_j &= \lambda_j^2 \varphi_j, & &\text{on } M, \\
\langle \varphi_j, \varphi_k \rangle &= \delta_{jk} \\
B \varphi_j &= 0 & &\text{on } \partial M.
\end{align*}
$$

Here, $\langle f, g \rangle = \int_M f \bar{g} dV$ is the $L^2(M)$ inner product with respect to the induced Riemannian volume form $dV$, and where $B$ is the boundary operator, either $B \varphi = \varphi|_{\partial M}$ in the Dirichlet case or $B \varphi = \partial_{ \nu} \varphi|_{\partial M}$ in the Neumann case.

We introduce a hypersurface $H \subset M$, which we assume to be orientable, embedded, and separating in the sense that

$$M \setminus H = M_+ \cup M_-$$

where $M_\pm$ are domains with boundary in $M$. This is not a restrictive assumption since we can arrange that any hypersurface is part of the boundary of a domain. This note is concerned with small-scale (2-microlocal) control estimates for the eigenfunction Cauchy data along $H$, where

$$CD(\lambda) := (\varphi_\lambda|_H, \lambda^{-1} \partial_ \nu \varphi_\lambda|_H).$$

Specifically, we present two independent but closely-related results here.

Our main result deals with $L^2$-restriction bounds for the normalized Neumann data $\lambda^{-1} \partial_\nu \varphi_\lambda|_H$.

**Theorem 1.** Suppose $H \subset M$ is a smooth, embedded orientable separating hypersurface and assume that $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Let $\{\varphi_{\lambda_j}\}_{j=1}^\infty$ denote the $L^2$-orthonormalized Laplace eigenfunctions on $M$. Then,

$$\|\lambda_j^{-1} \partial_\nu \varphi_{\lambda_j} \|_{L^2(H)} = O(1).$$

Theorem 1 generalizes a result of Hassell and Tao [HT] for boundary traces of Dirichlet eigenfunctions to arbitrary interior hypersurfaces. We note that the
universal $L^2$-restriction upper bound in Theorem 1 for the normalized Neumann data $\lambda^{-1}\partial_\nu \varphi_\lambda|_H$ is sharp and is substantially better than for the corresponding Dirichlet data $\varphi_\lambda|_H$ which, by [BGT], is only $O(\lambda^{1/2})$. The latter estimate is also sharp.

We remark also that we have been informed that Melissa Tacy has also obtained an independent proof of Theorem 1 using different methods in a current work in progress.

Theorem 1 follows from a Rellich commutator argument together with small-scale mass estimates for the tangential and exterior frequency components of the eigenfunction restrictions. Since the Rellich identity is of independent interest (see Remark 1), we have decided to state the Rellich identity in the greatest generality possible.

Our second result then is a 2-microlocal positive-commutator (Rellich) identity (see Theorem 2 below) which controls the $L^2$ restriction of the normalized eigenfunction Cauchy data along $H$ in terms of interior mass on $h^3$-neighbourhoods of the hypersurface. This somewhat technical result applies to any sequence of eigenfunctions (quantum ergodic or not).

In the following we let the semiclassical parameter $h \in \{\lambda_j^{-1}\}_{j=1}^\infty$ and rescale the Laplacian to the semiclassical operator $P(h) = -h^2\Delta_P$. We abuse notation somewhat and write $\varphi_h = \varphi_\lambda$ for the eigenfunction with eigenvalue $\lambda^2 = h^{-2}$. To state the relevant commutator estimate, we introduce various $h$-pseudodifferential cutoffs suppressing for the moment some of the technical details. Let $\chi \in C^\infty_0(\mathbb{R}; [0, 1])$ with $\chi(u) = 1$ for $|u| \leq 1/2$ and $\chi(u) = 0$ for $|u| > 1$, $\chi_- \in C^\infty(\mathbb{R})$ with $\chi_-(u) = 1$ when $u < -1$ and $\chi_+ \in C^\infty(\mathbb{R})$ with $\chi_+(u) = 1$ when $u > 1$. In addition we require that

$$\chi_-(u) + \chi(u) + \chi_+(u) = 1; \ u \in \mathbb{R}.$$  

Let $R(x', \xi') = \sigma(-\Delta_{H})(x', \xi')$ be the principal symbol of the induced hypersurface Laplacian $\Delta_H^p : C^\infty(H) \to C^\infty(H)$ and consider the decomposition of $T^*H$ into 3 pieces given by the radial cutoffs $\chi_{in}, \chi_{tan}, \chi_{out} \in C^\infty(\mathbb{R}; [0, 1])$ with $\chi_{in}(x', \xi') = \chi_-(R(x', \xi') - 1)$, $\chi_{tan}(x', \xi') = \chi(R(x', \xi') - 1)$ and $\chi_{out}(x', \xi') = \chi_+(R(x', \xi') - 1)$. Clearly, $\chi_{in} \subset B^*_{++}H$, $\chi_{out} \in T^*H - \overline{B^*_{++}H}$ and in addition,

$$\chi_{in}(x', \xi') + \chi_{tan}(x', \xi') + \chi_{out}(x', \xi') = 1; \ (x', \xi') \in T^*H.$$  

For any $\delta \in [0, 1)$ we also define the rescaled cutoff functions by $(\chi_{in})_{h, \delta}(x, \xi) = \chi_-(h^{-\delta}(R(x', \xi') - 1)$, $(\chi_{tan})_{h, \delta}(x, \xi) = \chi(h^{-\delta}(R(x', \xi') - 1)$, and $(\chi_{out})_{h, \delta}(x, \xi) = \chi_+(h^{-\delta}(R(x', \xi') - 1)).$

We denote the corresponding $h$-Weyl pseudodifferential partition of unity by $(\chi_{tan})_{h, \delta} = \text{Op}^w_{\infty}(\chi_{tan})_{h, \delta}$ and similarly for $(\chi_{out})_{h, \delta}$ and $(\chi_{in})_{h, \delta}$ (see Figure 1). In the following, we denote the canonical restriction map by $\gamma_H : C^\infty(M) \to C^\infty(H)$ and the corresponding eigenfunction restriction by $\varphi_h^H := \gamma_H \varphi_h \in C^\infty(H)$.

The main technical step in the proof of Theorem 1 is the following small-scale semiclassical Rellich identity. In keeping the notation consistent with that established in the literature, we define

$$\varphi_h^H = \varphi_h|_H, \quad \varphi_h^{H, \nu} = hD_\nu \varphi_h|_H,$$

where $\nu$ is the outward pointing unit normal vector and $D = i^{-1}\partial$ as usual.

**Theorem 2.** Suppose $H \subset M$ is a smooth, embedded orientable separating hypersurface with interior component $M_-$ and assume $H \cap \partial M = \emptyset$ if $\partial M \neq \emptyset$. Let
\{ \varphi_h : h \in \{ \lambda_j^{-1} \} \} \text{ denote the } L^2\text{-orthonormalized Laplace eigenfunctions on } M. \text{ Then for any } \delta \in [0,1) \text{ and } h \in (0,h_0(\delta)) \text{ sufficiently small, we have}

\begin{align*}
\langle (1 + h^2 \Delta_H) (\chi_{\text{in,\,tan}})_h^w \varphi_h, \varphi_h \rangle_{L^2(H)} &+ \langle (\chi_{\text{in,\,tan}})_h^w \varphi_h^H, \varphi_h^H \rangle_{L^2(H)} \\
= \langle b^w_{\text{in,\,tan}}(x, h D_x) \varphi_h, \varphi_h \rangle_{L^2(M_-)} + O(h^\infty).
\end{align*}

Here

\begin{align*}
b^w_{\text{in,\,tan}}(x, h D_x) &= b^w_{1,\text{in,\,tan}}(x, h D_x) + b^w_{2,\text{in,\,tan}}(x, h D_x) \\
&= \frac{i}{h} \left[ h^2 D_{x_n}, \chi(x_n) h D_{x_n} \chi_{\text{in,\,tan}}w \left( R(x', x_n = 0, \xi') - \frac{1}{h^\delta} \right) \right] \\
&\quad + \frac{i}{h} \left[ R(x', x_n, h D_{x'}), \chi(x_n) h D_{x_n} \chi_{\text{in,\,tan}}w \left( R(x', 0, \xi') - \frac{1}{h^\delta} \right) \right] \\
&\quad + O(h^{1-\delta})_{L^2 \rightarrow L^2},
\end{align*}

(1.2)

with

\begin{align*}
b^w_{1,\text{in,\,tan}} &\in S^0_{\Sigma H, \delta} \quad \text{and} \quad b^w_{2,\text{in,\,tan}} \in S^\delta_{\Sigma H, \delta}.
\end{align*}

The operators \((\chi_{\text{\,tan}})_h^w\) are in standard operator classes when \(\delta \in [0,1/2)\), but for \(\delta \in [1/2,1)\) they belong to a second microlocal class of semiclassical pseudodifferential operators in \(Op_h^w(S^0_{\Sigma H, \delta})\) (see Definition 2.1). The estimates in Theorem
2 in the latter case are more subtle. To state the stronger result, we need to introduce an appropriate class of \( h \)-pseudodifferential operators \( \text{Op}_h^w \Sigma \cup H (S^0) \) second microlocalized along \( S^* H \) and prove a Rellich-type bound for these operators. The appropriate 2-microlocal calculus is treated in section 2.2.2. The closely related calculus in [SjZw1, SjZw2] is carried out in the case where \( \Sigma \subset T^* M \) is a hypersurface, but because Fermi coordinates separate base variables near \( H \) one can carry out an effective 2-microlocalization along the codimension 2 submanifold of \( T^* M \) given by

\[
\Sigma_H := (S^* H + N^*_H) \cap T^*_H M
\]

Here, in terms of Fermi coordinates \((x_n, x') \) near \( H \), \( T^*_H M = \{(x', x_n = 0; \xi', \xi_n)\} \), \( N^*_H = \{(x'; x_n = 0, \xi' = 0, \xi_n); |\xi_n| < \epsilon\} \) and \( \Sigma_H = \{(x', x_n = 0; \xi', \xi_n); R(x', \xi') = 1, |\xi_n| < \epsilon\} \).

**Remark.** The results here make no dynamical assumptions, but when \((M, g)\) is ergodic and \( \delta = 0 \), the special case of Theorem 2 yields the quantum ergodic restriction theorem (QER) in [CTZ] for quantum ergodic sequences of interior eigenfunctions. It is an interesting problem to determine whether Theorem 2 yields a "small-scale" quantum ergodic restriction theorem when \( \delta \in [0, 1) \). Specifically, given an oriented separating hypersurface \( H \subset M \), we say that the sequence \((\varphi_{h_j})_{j \in S}\) of eigenfunctions is \( \delta \text{-quantum ergodic} \) (abbreviated \( \delta \text{-QE} \)) near \( H \) if for any symbol \( a(x, \xi; h) \in \mathcal{S}^0_{\Sigma_H, \delta}(T^* M) \),

\[
\langle \text{Op}_h^w (a(x, \xi; h_j)) \varphi_{h_j}, \varphi_{h_j} \rangle \sim_{h_j \to 0} \int_{S^* M} a(x, \xi, h_j) \mu_L.
\]

Given a subsequence \( \{\varphi_{h_k}\}_{k \in S} \) that is \( \delta \text{-QE} \) near \( H \), it is of interest to determine under what conditions the Cauchy data \( \{CD(\varphi_{h_k})\}_{k \in S} \) has the \( \delta \text{-quantum ergodic restriction} \) \( (\delta \text{-QER}) \) property on \( H \), in the sense that for any \( a(x', \xi'; h_k) \in \mathcal{S}^0_{\Sigma_H, \delta}(T^* H) \)

\[
\langle (\text{Id} + h_k^2 H) a^w_{\nu, \nu'} \varphi_{h_k}, \varphi_{h_k} \rangle + \langle a^w_{\nu, \nu'} H a^w_{\nu, \nu'} \varphi_{h_k}, \varphi_{h_k} \rangle \sim_{h_k \to 0} 2 \int_{B \cdot H} a(x', \xi'; h_k) \sqrt{1 - |\xi'|^2} dx'd\xi'.
\]

(1.3)

Following the argument in [CTZ] (see also [Bur]), the general commutator identity in Theorem 2 will play a central role in extending the QER results of [CTZ] to small-scale symbols in (1.3). We will explore this in a forthcoming work [CTZ2].

**2. Second-microlocalized pseudodifferential operators**

We collect and briefly review here the requisite semiclassical analysis. This includes the various 2-microlocal symbol classes and the corresponding pseudodifferential operator calculus. The material here is a special case of more general two-parameter calculus developed in [SjZw1, SjZw2]. Since our interest lies in establishing \( L^2 \)-restriction bounds for eigenfunctions along a hypersurface \( H \subset M \), we need only consider ambient symbols supported in an \( \epsilon > 0 \) neighbourhood of \( H \), where \( \epsilon > 0 \) is arbitrary small. Thus, we introduce Fermi coordinates \( x = (x', x_n) \) near \( H \) with \( x = \exp_{\nu, \nu'}(x_n \nu_{\nu'}) \) with \( \nu_{\nu'} \) an exterior unit normal to \( H \). Since by assumption \( H \) is orientable, this is well-defined. In Fermi coordinates, \( H = \{x_n = 0\} \) and it is convenient to define our symbols in terms of these coordinates. We do so without further comment.
2.1. Homogeneous and semiclassical symbol classes. We collect for future reference a brief review of the standard symbol classes and corresponding pseudodifferential operators used later on (see also [Zw, Section 4.4]). The more subtle microlocal semiclassical analysis is treated in 2.2.2.

The standard homogeneous symbol spaces that are relevant here are
\[ S^m_{p,b}(T^*M) \]
(2.1) \[ = \{ a(x,ξ) \in C^\infty(T^*M - 0); |\partial_x^\alpha \partial_\xi^\beta a(x,ξ)| = O_{\alpha,\beta}(|ξ|^{-m-\rho|\alpha|+\delta|\beta|}), \rho > \delta \} \]
As for the semiclassical symbols, the relevant symbol classes for our purposes are
\[ S^m_{cl}(T^*M \times (0,h_0]) = \{ a(x,ξ;h) \in C^\infty(T^*M \times (0,h_0]); |\partial_x^\alpha \partial_\xi^\beta a| = O_{\alpha,\beta}(h^{-m}h^{-\delta(|\alpha|+|\beta|)}⟨ξ⟩^{-\infty}) \}, \]
with \( \delta \in [0,1) \). Since both the eigenfunctions \( \varphi_{h} \) and their restrictions \( u_h = \varphi_{h}|H \) have compact \( h \)-wavefront sets (see for example [Zw, Section 8.4] and Section 5 below) we are interested here in only the case where the \( ξ \) variables are in a compact set. Consequently, the semiclassical symbol classes \( S^m_\delta \) are most relevant. In the special case where \( \delta = 0 \),
\[ S^m_0(T^*M \times (0,h_0]) = \{ a \in C^\infty(T^*M \times (0,h_0]); |\partial_x^\alpha \partial_\xi^\beta a| = O_{\alpha,\beta}(h^{-m}⟨ξ⟩^{-\infty}) \}. \]
When the context is clear, we sometimes just write \( S^m_\delta \) instead of \( S^m(T^*M \times (0,h_0]) \). The case where \( \delta = 0 \) is sometimes denoted by \( S^m(1) \) in the literature.

The corresponding \( h \)-Weyl pseudodifferential operators have Schwartz kernels that are sums of the local integrals of the form
\[ Op^w_h(a)(x,y) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{i(x-y,ξ)/h}a(x,ξ,h)dx. \]
We will use \( Op^w_h(a), a^w(x,hD_x) \) and \( a^w(x,hD_x) \) interchangeably to denote \( h \)-Weyl quantizations of \( a(x,ξ,h) \) since each has its advantages. It is standard that for \( a \in S^{m_1,k_1}_{cl}, b \in S^{m_2,k_2}_{cl} \),
\[ a^w(x,hD_x) \circ b^w(x,hD_x) = e^{ih\sigma(D_x,D_ξ,D_\psi,D_\eta)/2}a(x,ξ)b(y,η)|_{y=x,η=ξ} \]
with \( c(x,ξ,h) = a(x,ξ,h)\# b(x,ξ,h) \) and \( σ(x,ξ,y,η) = yξ - xη \). Similarly, for \( a \in S^m_{cl,b} \), \( b \in S^m_{cl} \) with \( \delta \in [0,1/2) \),
\[ a^w(x,hD_x) \circ b^w(x,hD_x) = c^w(x,hD_x) \]
with \( c(x,ξ,h) = a(x,ξ,h)\# b(x,ξ,h) \).
Since eigenfunctions (and their restrictions) have compact \( h \)-wavefront, it is the algebra \( Op^w_h(S^m_\delta) \) that is most relevant here. We point out that for \( a^w(x,hD_x) \in Op^w_h(S^m_\delta), 0 \leq \delta < 1/2, \) with \( a(x,ξ,h) \geq 0 \), one also has the sharp Gårding inequality
\[ a^w(x, hD_x) \geq -Ch^{1-2\delta} \] (in the \( L^2 \) sense) and indeed the sharper Fefferman-Phong inequality
\[ a^w(x, hD_x) \geq -Ch^{2-4\delta} \]
also holds [Zw, Section 4.7].

2.2. Semiclassical second-microlocal pseudodifferential cutoffs: microlocal decomposition.

2.2.1. Fermi normal coordinates near \( H \). From now on, we let \( x = (x', x_n) \) be Fermi normal coordinates in a small tubular neighbourhood \( H(\epsilon) \) of \( H \) defined near a point \( x_0 \in H \). In these coordinates we can locally write
\[ H(\epsilon) := \{(x', x_n) \in U \times \mathbb{R}, |x_n| < \epsilon \}. \]
Here \( U \subset \mathbb{R}^{n-1} \) is a coordinate chart containing \( x_0 \in H \) and \( \epsilon > 0 \) is arbitrarily small but for the moment, fixed. We let \( \chi \in C_0^\infty(\mathbb{R}) \) be a cutoff with \( \chi(x) = 0 \) for \( |x| \geq 1 \) and \( \chi(x) = 1 \) for \( |x| \leq 1/2 \). Moreover, in terms of the normal coordinates,
\[ -h^2 \Delta_g = \frac{1}{g(x)} hD_{x_n}g(x)hD_{x_n} + R(x_n, x', hD_x) \]
where \( R \) is a second-order \( h \)-differential operator along \( H \) with coefficients that depend on \( x_n \), and \( R(0, x', hD_x) \) is the induced tangential semiclassical Laplacian, \( \Delta_H \), on \( H \). Consequently, at the level of symbols,
\[ \sigma(-h^2 \Delta_g)(x, \xi) = |\xi|^2 \gamma = \xi_n^2 + R(x', x_n, \xi'), \]
where,
\[ \sigma(-h^2 \Delta_H)(x', \xi') = R(x', 0, \xi'). \]

Let \( \gamma_H : C^0(M) \to C^0(H) \) be the restriction operator \( \gamma_H(f) = f|_H \). The adjoint \( \gamma_H^* : C^0(H) \to C^0(M) \) is then given by
\[ \gamma_H^*(g) = g \cdot \delta_H. \]

When there is no risk for confusion we write \( \varphi^H_h = \gamma_H \varphi_h \) and similarly \( \varphi^{H, \nu}_h = -ih\gamma_H \partial_{x_n} \varphi_h \).

To describe the relevant 2-microlocal \( h \) pseudodifferential operators, we first describe the geometry. For \( 0 < \delta < 1 \), the second microlocalization along \( S^*H \subset T^*M \) is achieved in several steps.

The first step involves cutting off to a small neighbourhood of the hypersurface
\[ \Sigma_1 := \{(x, \xi) \in H(\epsilon); x_n = 0\} = T^*_H M, \]
using a 2-microlocal cutoff \( a_1(x_n; h) \) in the normal variable \( x_n \).

Next, we make a full 2-microlocal cutoff to scale \( h^\delta \) around the hypersurface \( \Sigma_2 \subset T^*M \) given by
\[ \Sigma_2 := \{(x, \xi) \in H(\epsilon); R(x', x_n = 0, \xi') = 1\} \]
where we note that \( R(x', 0, \xi') \) involves coordinates \( (x', \xi') \) complementary to \( (x_n, \xi_n) \) and so, in particular,
\[ \{x_n, R(x', 0, \xi')\} = 0. \]
This allows us to simultaneously 2-microlocalize to the codimension 2 submanifold \( \Sigma_H := \Sigma_1 \cap \Sigma_2 \subset T^*M \). In some applications, we localize to \( |x_n|, |\xi_n| \leq h^{\delta/2} \), while in others we only localize in an \( h \)-independent neighbourhood in \( x_n \) of the form \( |x_n| \leq h^{\delta} \). However, the symbols are quite explicit, so no confusion should arise.
The relevant $h$-neighbourhoods of $\Sigma_j$, $j = 1, 2$ are $\Sigma_1(h) = \{(x, \xi) \in H(\epsilon), |x_n| \leq h^{\delta/2}\}$ and $\Sigma_2(h) = \{(x, \xi) \in H(\epsilon), |R(x', 0, \xi') - 1| \leq h^{\delta}\}$. Finally, the eigenfunction mass 2-microlocalization is concentrated in the set $\Sigma_H(h) = \Sigma_1(h) \cap \Sigma_2(h)$.

Clearly, it follows that as a subset of $T^*M$, the intersection $\Sigma_1 \cap \Sigma_2 \subset T^*M$ is the codimension-two submanifold $\Sigma_H \subset \mathbb{R}_{\nu} \cdot S^*H$ given by

$$\Sigma_H = \{(x', x_n = 0; \xi', \xi_n) \in T^*M; R(x', x_n = 0, \xi') = 1, |\xi_n| < \epsilon\}.\tag{2.6}$$

This is a subset of the total space of the normal line bundle over $S^*H$ with $(x', \xi'; v) \in \Sigma_H$ if and only if $(x', \xi') \in S^*H$ and $v = \nu_{x'}$ with $|a| < \epsilon$ and $\nu_{x'}$ the unit exterior normal at $x' \in H$. Clearly, $\Sigma_H \subset T^*M$ is a submanifold of codimension two and is the total space of a trivial line bundle over the glancing set $S^*H$, where the trivialization is given by Fermi coordinates $(x', x_n) \in H(\epsilon)$ with $x = \exp_{x'}(x_n\nu_{x'})$.

2.2.2. Semiclassical pseudodifferential operators second microlocalized along $\Sigma_H$.

We introduce here the relevant 2-microlocal algebra of $h$-pseudodifferential operators localized on very small scales $\sim h^{\delta}$ where $0 < \delta < 1$ that will be used in the proof of our control estimate in Theorem 2. When $\delta \in (0, 1/2)$ the pseudodifferential calculus is well-known [Zw, Chapter 4] but for $\delta > 1/2$ the construction is more subtle. The relevant calculus has been developed in very general framework by Sjöstrand and Zworski [SjZw1, SjZw2] to which we refer the reader for further details. Since a rather simple special case of their calculus will suffice for our purposes, we will attempt to keep the argument fairly self-contained.

**Definition 2.1.** Let $H \subset M$ be a hypersurface. We say that a semiclassical symbol $b$ is 2-microlocalized along $\Sigma_H$ and write $b \in S_{\Sigma_H, \delta}^m(T^*M \times (0, h_0))$ provided there exists $\chi \in C_0^\infty(\mathbb{R})$, $a_1(x_n, \xi_n; h) \in S_{h^{\delta/2}}^0(T^*H(\epsilon) \times (0, h_0))$ such that

$$b(x, \xi; h) = h^{-m} a_1(x_n, \xi_n; h) \cdot \chi \left(\frac{R(x', x_n = 0, \xi') - 1}{h^{\delta}}\right), \quad 0 \leq \delta < 1$$

for all $(x, \xi) \in T^*M$.

**Remark.** We note that in the specific case where

$$a_1(x_n, \xi_n; h) = \chi(h^{-\delta}x_n) \cdot \chi(\xi_n), \quad \chi \in C_0^\infty(\mathbb{R}),$$

by rescaling in phase space $(x_n, \xi_n) \mapsto (h^{\delta/2}x_n, h^{-\delta/2}\xi_n)$ produces a symbol $a_1 \in S_{h^{\delta/2}}^0$. In other words, since we are always using the Weyl calculus, it is sufficient that $(\partial_{x_n} \partial_{\xi_n})^k a_1 = O(h^{-\delta})$ to be in a good calculus.

We will need the following proposition (see also [SjZw1, SjZw2]).

**Proposition 2.2.** Given $a^w(x, hD_x) \in Op_h(S_{\Sigma_H, \delta}^{m_1})$ and $b^w(x, hD_x) \in Op_h(S_{\Sigma_H, \delta}^{m_2})$ it follows that

$$a^w(x, hD_x) \circ b^w(x, hD_x) = c^w(x, hD_x) \in Op_h(S_{\Sigma_H, \delta}^{m_1 + m_2})$$

with

$$c(x, \xi; h) = a(x, \xi, h) \# b(x, \xi, h).$$

**Proof.** Since

$$d\xi' R(x', 0, \xi') \geq C|\xi'|, \quad C > 0$$

near $\{R(x', 0, \xi') = 1\}$, as in [SjZw1, SjZw2], the proof hinges on the following real-principal type quantum normal form construction given in the following Lemma.
Lemma 2.3. Let \((x_0, \xi_0) \in \Sigma_H\) and let \(U \subset T^*H\) be a sufficiently small open neighbourhood of \((x_0, \xi_0)\). For \(U\) small, there exists \(V \subset T^*\mathbb{R}^{n-1}\) open together with a canonical transformation

\[ \kappa_F : (U, (x_0, \xi_0)) \to (V, (0, 0)); \quad \kappa_F(x', \xi') = (y', \eta') \]

and corresponding \(h\)-Fourier integral operators \(F(h) : C_0^\infty(U) \to C_0^\infty(V)\) such that

\[ (i) \quad F(h)^* \circ \chi_h^w \left( \frac{R(x', 0, \xi') - 1}{h^\delta} \right) \circ F(h) = U \times V \chi_h^w \left( \frac{\eta'}{h^\delta}; h \right), \]

\[ (ii) \quad F(h)^* \circ F(h) = U \times V \text{Id}, \]

with \(\chi \in C_0^\infty(\mathbb{R}^{n-1}; (0, 0) \times (0, h_0])\). Here, \(A(h) = U \times V B(h)\) denotes \(h\)-microlocal equivalence on \(U \times V \subset T^*H \times T^*\mathbb{R}^{n-1}\).

Given Lemma 2.3 one reduces the proof of the proposition to operators in normal form. In the conormal variables \((x_n, \xi_n)\) the composition formula is standard since symbols are in the standard \(S^0_{1,2}\)-classes with \(\delta/2 < 1/2\). Since symbols in \(S^0_{1,2}\) are separable, it suffices to assume that \(a(x, \xi; h) = \chi_1(h^{-\delta}(R(x', 0, \xi') - 1))\) and \(b(x, \xi; h) = \chi_2(h^{-\delta}(R(x', 0, \xi') - 1))\) with \(\chi_j \in C_0^\infty(\mathbb{R}); j = 1, 2\). Let \(\psi_{U_j} \times \psi_{V_j} \in Op_h(S^0_0); j = 1, ..., N_0\) be an \(h\)-microlocal partition of unity subordinate to a covering of the supports of \(a\) and \(b\) in \(T^*H\) by open sets \(U_j; j = 1, ..., N_0\) such that on each \(U_j\) Lemma 2.3 holds with \(h\)-FIO \(F(h)\) with \(WF_h(F(h)) \subset U_j \times V_j\). From Lemma 2.3,

\[ a_h^w \circ b_h^w = \sum_{j=1}^{N_0} \psi_{U_j}^w F(h) \circ [F(h)^* a_h^w F(h)] \circ [F(h)^* b_h^w F(h)] \circ F(h)^* \psi_{V_j}^w + R(h), \]

where \(R(h) \in Op_h(S_0^{-\infty})\). For the inner model operators one simply rescales the fiber variables \((y', \eta'/h^\delta) \to (y'/h^{\delta/2}, \eta'/h^{\delta/2})\) and computes the composition in the model normal coordinates. The result is that in each chart

\[ [F(h)^* a_h^w F(h)] \circ [F(h)^* b_h^w F(h)] = (\chi_1(h^{-\delta} \eta_1^1) \# \chi_2(h^{-\delta} \eta_1^2))^w + \mathcal{O}(h^\infty)_{L^2 \to L^2}\]

\[ = (2\pi h)^{-(n-1)} \int_{\mathbb{R}^{n-1}} e^{i(x' - y' \cdot \eta')} h \psi_{V_j}(\frac{2^{\delta/2} y'}{2h^{\delta/2}}, h^{\delta/2} \eta')(\chi_1 \# \chi_2)(h^{-\delta/2} \eta_1^2) d\eta' + \mathcal{O}(h^\infty)_{L^2 \to L^2}. \]

The \(\#\) product expansion is computed as usual in the \(h^{-\delta/2}\) calculus and then rescaled, which is particularly simple for our special choice of operators:

\[ (\chi_1 \# \chi_2)(h^{-\delta} \eta_1^2) = \sum_{j=0}^{\infty} h^j (D_{x_n}^j D_{\xi_n} - D_y D_{\xi_j})^j [\chi_1(h^{-\delta} \eta_1^2) \cdot \chi_2(h^{-\delta} \xi_1^j)] |\xi' = \eta'| \]

\[ = \chi_1(h^{-\delta} \eta_1^2) \cdot \chi_2(h^{-\delta} \eta_1^2) + R(y', \eta'; h), \]

where, \(R(y', \eta'; h) \in S_0^{-\infty}(T^*\mathbb{R}^{n-1}). \)

\(\square\)
Remark. We note that $L^2$ boundedness for $a^w_h \in \text{Op}_h(S^0_{0,\nu})$ with $\nu \in [0,1)$ is clear by passing to normal form. Since $F(h)^*F(h) \cong_{U_j \times V_j} \text{Id}$, (2.9)
\[
\|a^w(x, hD_x)\|_{L^2 \to L^2} \leq \sum_{j=1}^{N_0} \|\chi_h^w(h^{-\delta} \eta'_1)\psi_{V,j}(y', \eta')\|_{L^2 \to L^2}
\]
\[
= \sum_{j=1}^{N_0} \|\chi(h^{1-\delta} \eta'_1)\psi_{V,j}(y', h\eta')\|_{L^2 \to L^2}
\]
\[
\leq \sum_{|\alpha| \leq 2n+1} \|\partial^\alpha \chi(h^{1-\delta} \eta'_1)\psi_{V,j}(y', h\eta')\|_{L^\infty}
\]
\[
\leq \|\chi(h^{1-\delta} \eta'_1)\psi_{V,j}(y', h\eta')\|_{L^\infty} + O(h^{1-\delta}).
\]
This follows by rescaling $\eta' \mapsto h\eta'$ in the fiber variables and then applying the Calderon-Vaillancourt theorem, together with the trivial estimates on the derivatives of the functions $\chi$ and $\psi_{V,j}$.

3. Rellich identity for $h$-pseudodifferential operators
2-microlocalized along $\Sigma = S^*H$: proof of Theorem 2

Our main theorem relies on a Rellich identity which holds for $a^w(x, hD_x) \in \text{Op}_h(S^0_{0,\nu})$.

Proof of Theorem 2. We first assume $H = \partial M_-$ where $M_-$ is a smooth open submanifold of $M$. This assumption is to be able to use Green’s formula, but since we are allowed a cutoff to a subset of $H$, it is purely technical. We use a Rellich type identity to write the integral of a commutator over $M_-$ as a sum of integrals over the boundary. The argument is partially motivated by Burq’s proof of boundary quantum ergodicity (ie. the case $H = \partial M$) in [Bur] and our earlier joint paper [CTZ].

First, we choose $\psi \in C^\infty_c(\mathbb{R}), \psi(s) = 1$ for $|s| \leq 2$, and we put
\[
a^w(x, hD_x) := \chi(x_n) \psi(hD_{x_n}) hD_{x_n} \chi(n)_{h,\delta}(x', hD_{x'}) \in \text{Op}_h(S^0_{0,\nu})\).
\]
where $\chi(u) \equiv 1$ for $|u| \leq 1/2$ and $\chi(u) \equiv 0$ for $|u| > 1$ as in the introduction. Then, a straightforward application of Green’s formula and using that $(-\hbar^2 \Delta_g - 1)\varphi_h = 0$ yields the Rellich formula
\[
i \frac{\hbar}{i} \int_{M_-} [-\hbar^2 \Delta_g, a^w(x, hD_x)] \varphi_h(x) \overline{\varphi_h(x)} \, dx
\]
\[
= -i \int_H h\partial_{x_n} a^w(x', x_n, hD_x) \varphi_h|_{H \overline{\varphi_h}} \, d\sigma_H
\]
\[
+ i \int_H a^w(x', x_n, hD_x) \varphi_h|_{H \overline{h\partial_{x_n} \varphi_h}} \, d\sigma_H
\]
\[
= \int_H hD_{x_n} a^w(x', x_n, hD_x) \varphi_h|_{H \overline{hD_{x_n} \varphi_h}} \, d\sigma_H
\]
\[
+ \int_H a^w(x', x_n, hD_x) \varphi_h|_{H \overline{hD_{x_n} \varphi_h}} \, d\sigma_H.
\]
In the following we continue to use the notation $\varphi^H_h := \varphi_h|_H$ and $\varphi^{H,\nu}_h := hD_{\nu} \varphi_h|_H$. We begin by computing the RHS of (3.1).
Since $\chi(0) = 1$ and $[\psi^w(hD_{x_n}), (\chi_{in})^w_{h,\delta}(x', hD_{x'})] = 0$, it follows that the second term on the RHS of (3.1) is just

\begin{equation}
(3.2)
\langle (\chi_{in})^w_{h,\delta}(x', hD_{x'}) \psi^w(hD_{x_n}) \varphi^H, \varphi^H \rangle + O(h^\infty).
\end{equation}

As for the first term on RHS of (3.1) we get

\begin{equation}
(3.3)
\int_H hD_{x_n} \left( \chi(x_n) \psi^w(hD_{x_n}) hD_{x_n} \chi_{in})^w_{h,\delta}(x', hD_{x'}) \varphi_h \right) \bigg|_{x_n=0} \cdot \varphi_h|_{x_n=0} d\sigma_H
\end{equation}

\begin{equation}
(3.4)
= \int_H \left[ \chi(x_n) \psi^w(hD_{x_n}) (\chi_{in})^w_{h,\delta}(x', hD_{x'}) (hD_{x_n})^2 \varphi_h
- i\hbar \chi'(x_n) \psi^w(hD_{x_n}) (\chi_{in})^w_{h,\delta}(x', hD_{x'}) hD_{x_n} \varphi_h \right] \bigg|_{x_n=0} \cdot \varphi_h|_{x_n=0} d\sigma_H
\end{equation}

\begin{equation}
= \int_H \left[ a(x, hD_x)(1 - R(x_n, x', hD_{x'}))\varphi_h \right] \bigg|_{x_n=0} \cdot \varphi_h|_{x_n=0} d\sigma_H,
\end{equation}

since $\chi'(0) = 0$. Moreover, $((hD_{x_n})^2 + R(x, hD_{x'}))\varphi_h = \varphi_h$ in Fermi coordinates and one can add an energy cutoff (modulo $O(h^\infty)$-error) above to the region \{(x', \xi); \xi_n^2 + R(x_n, 0, x', \xi') = 1 + O(h^\delta)\}. But then since $R(x_n, 0, x', \xi') = \sigma(-h^2\Delta_H)(x', \xi') \geq 0$, we have $|\xi_n| \leq 1$ on this set and so the frequency cutoff $\psi^w(hD_{x_n}) = 1$. Consequently, modulo $O(h^\infty)$-error, (3.3) is equal to

\begin{equation}
(3.5)
= \langle (\chi_{in})^w_{h,\delta}(x', hD_{x'}) (1 + h^2\Delta_H)\varphi^H, \varphi^H \rangle.
\end{equation}

Thus, from (3.2), (3.5) and (3.1), it follows that

\begin{equation}
(3.6)
\langle (\chi_{in})^w_{h,\delta}(x', hD_{x'}) \varphi^H, \varphi^H \rangle_{L^2(H)}
+ \langle (\chi_{in})^w_{h,\delta}(x', hD_{x'}) (1 + h^2\Delta_H)\varphi^H, \varphi^H \rangle_{L^2(H)}
= \langle b^w(x, hD_x)\varphi_h, \varphi_h \rangle_{L^2(M_-)} + O(h^\infty)
\end{equation}

where,

\begin{equation}
(3.7)
b^w(x, hD_x) = \frac{i}{\hbar} [-h^2\Delta_g, a^w(x, hD_x)] \in Op_h^w(S^\delta_{\Sigma_H, \delta}).
\end{equation}

It follows that

\begin{equation}
(3.8)
b^w(x, hD_x)
= \frac{i}{\hbar} \left[ h^2D^2_{x_n} \psi^w_{h}(hD_{x_n}), \chi(x_n) hD_{x_n} \chi_{in} \left( \frac{R(x', 0, \xi') - 1}{\hbar^2} \right) \right]
+ \frac{i}{\hbar} \left[ R(x', x_n, hD_{x'}), \chi(x_n) hD_{x_n} \chi_{in} \left( \frac{R(x', 0, \xi') - 1}{\hbar^2} \right) \right]
+ O(h^\infty)_{L^2 \to L^2} =: b^w_1(x, hD_x) + b^w_2(x, hD_x).
\end{equation}

Clearly,

\begin{equation}
(3.9)
b^w_1(x, hD_x)
= \frac{i}{\hbar} \left[ h^2D^2_{x_n} \psi^w_{h}(hD_{x_n}), \chi(x_n) hD_{x_n} \chi_{in} \left( \frac{R(x', 0, \xi') - 1}{\hbar^2} \right) \right]
\in Op_h^w(S^\delta_{\Sigma_H, \delta})
\end{equation}
with
\[ \sigma(b_1)(x, \xi; h) = 2 \xi^2 \psi(\xi_n) \chi'(x_n) \chi_{in}(\frac{R(x', 0, \xi') - 1}{h^\delta}). \]

As for the second term,
\[ b^w_2(x, hD_x) = \frac{i}{h} [R(x_n, x', hD_x), \chi(x_n)hD_{x_n} \chi_{in}(\frac{R(x', 0, \xi') - 1}{h^\delta})] \in Op^w(h(S_{\Sigma_H}^\delta, \delta)). \]

Thus, \( b^w(x, hD_x) \in Op^w(h(S_{\Sigma_H}^\delta, \delta)) \) and by \( L^2 \)-boundedness (see (2.9)),
\[ b^w(x, hD_x) = b^w_1(x, hD_x) + \mathcal{O}(1)_{L^2 \to L^2}. \]

Finally, we simply note that exactly the same argument holds for \( \chi_{tan} \) replacing \( \chi_{in} \) in the above.

\[ \square \]

4. Estimating Interior terms

4.1. Dirichlet data. From Theorem 2 we have with \( b^w_{in} \in Op_{h}(S_{\Sigma_H}^\delta, \delta), \)
\[ \left\langle (1 + h^2 \nabla_H)(\chi_{in})^w_{h, \delta}, \varphi_h^H \right\rangle_{L^2(H)} + \left\langle (\chi_{in})_{h, \delta}^w, \varphi_h^H, \varphi_h^{H, \nu} \right\rangle_{L^2(H)} \]
\[ = \left\langle b^w_{in}(x, hD_x), \varphi_h^{H, \nu} \right\rangle_{L^2(M_-)} + \mathcal{O}(1). \]

We recall from Theorem 2 that \( b^w_{in}(x, hD_x) = b_{1, in}(x, hD_x) + b_{2, in}(x, hD_x) \) where
\[ b^w_{1, in}(x, hD_x) = b^w_{2, in}(x, hD_x) \in Op^w(h(S_{\Sigma_H}^\delta, \delta)) \] and
\[ b^w_{2, in}(x, hD_x) = \frac{i}{h} [R(x', x_n, hD_x), \chi(x_n)hD_{x_n} \chi_{in, tan}(\frac{R(x', 0, \xi') - 1}{h^\delta})] \in Op^w(h(S_{\Sigma_H}^\delta, \delta)). \]

Consequently, by \( L^2 \)-boundedness of \( b^w_{1, in}(x, hD_x), \)
\[ \left\langle b^w_{in}(x, hD_x) \varphi_h, \varphi_h \right\rangle_{L^2(M_-)} = \left\langle b^w_{2, in}(x, hD_x) \varphi_h, \varphi_h \right\rangle_{L^2(M_-)} + \mathcal{O}(1). \]

Even though \( b^w_{2, in}(x, hD_x) \in Op^w(h(S_{\Sigma_H}^\delta, \delta), \) it follows from our argument in section 6 (see Remark 6) that the matrix elements \( \left\langle b^w_{2, in}(x, hD_x) \varphi_h, \varphi_h \right\rangle_{L^2(M_-)} = \mathcal{O}(1) \) and consequently,
\[ \left\langle b^w_{in}(x, hD_x) \varphi_h, \varphi_h \right\rangle_{L^2(M_-)} = \mathcal{O}(1). \]

Moreover, since \( 1 - R(x', x_n = 0, \xi') \geq h^\delta \) for \( (x', \xi') \in \text{supp}(\chi_{in}), \) it follows by the sharp Garding inequality that
\[ \left\langle (1 + h^2 \nabla_H)(\chi_{in})_{h, \delta}^w, \varphi_h^H, \varphi_h^{H, \nu} \right\rangle_{L^2(H)} \geq h^\delta \left\langle (\chi_{in})_{h, \delta}^w, \varphi_h^H, \varphi_h^{H, \nu} \right\rangle_{L^2(H)}. \]

Since the normal derivative term on the LHS of (4.1) is non-negative and in view of the global restriction bounds in [BGT] one gets the interior Dirichlet bound
\[ \left\langle (\chi_{in})_{h, \delta}^w, \varphi_h^H, \varphi_h^{H, \nu} \right\rangle_{L^2(H)} = \min\{O_3(h^{-\delta}), O(h^{-1/2})\}. \]
4.2. Neumann data. In view of (4.5), it follows from the $O(1)$-bound for the RHS of (4.1) in (4.4) that for the Neumann data one has the better bound

$$\langle (\chi_{in})_{h,\delta}^{(H)}, \varphi_h \rangle_{L^2(H)} = O(1).$$

The estimate (4.7) is a consequence of the matrix bound in (4.4) which in turn follows from the proof of Theorem 1 in section 6.

5. Estimating the exterior terms

5.1. Exterior mass estimates on $O(1)$-scales. A key issue is mass concentration of eigenfunctions: Let $\varphi_{\lambda_j}$, $j = 1, 2, \ldots$ be an $L^2$ orthonormal basis of Laplace eigenfunctions on $(M, g)$ and $H \subset M$ a hypersurface. To begin we first discuss mass concentration of $\varphi_{H}^h$ on $B^*H$ on scales $\sim 1$ and then indicate the extension to small-scales $\sim h^\delta; \delta \in [0, 1]$ using 2-microlocal calculus in subsection 5.2.

Let $h \in \{\lambda_j^{-1}; j = 1, 2, \ldots\}$ be an arbitrary small number, let $\chi(x, \xi) \in C^\infty_0(T^*M)$ be equal to one on the annulus $A(\epsilon_0) = \{(x, \xi); (1 - \epsilon_0/10) < |\xi| < (1 + \epsilon_0/10)\}$ and with $\operatorname{supp} \chi \subset A(2\epsilon_0)$. Let $\tilde{\chi} \in C^\infty_0$ be another cutoff equal to one on $A(2\epsilon_0)$ and with $\operatorname{supp} \tilde{\chi} \subset A(4\epsilon_0)$. Consider the eigenfunction equation

$$(-h^2\Delta - 1)\varphi_h = 0.$$

Then, $P(h) := -h^2\Delta - 1$ is $h$-elliptic for $(x, \xi) \in T^*M - A(\epsilon)$. So, one can construct an $h$-microlocal parametrix with $Q(h) \in Op_h(S^0_{0,0})$ so that

$$(1 - \tilde{\chi}(h))Q(h)P(h)(1 - \tilde{\chi}(h))\varphi_h = (1 - \tilde{\chi}(h))\varphi_h + O(h^\infty).$$

Since $P(h)\varphi_h = 0$ and $\sigma([P(h), (1 - \tilde{\chi}(h))](x, \xi) = 0$ for $(x, \xi) \in \operatorname{supp}(1 - \tilde{\chi}(h))$, one gets the well-known concentration estimate

$$\| (1 - \tilde{\chi}(h))\varphi_h \|_{L^2} = O(h^\infty).$$

A similar argument with the derivatives $\partial_\xi^2 \varphi_h(x)$ combined with Sobolev embedding implies

$$\| (1 - \tilde{\chi}(h))\varphi_h \|_{C^k} = O(h^\infty),$$

and as a consequence

$$WF_h(\varphi_h) \subset S^*M.$$

Proposition 5.1. Let $H \subset M$ be a hypersurface and $\varphi_{H}^h := \varphi_h|_H = \gamma_H \varphi_h$ and $\varphi_{H,\nu}^h = \gamma_H(hD_{\nu}\varphi_h)$ as usual. Then,

$$WF_h(\varphi_{H}^h) \subset \overline{B^*H}$$

and

$$WF_h(\varphi_{H,\nu}^h) \subset \overline{B^*H}.$$
By a Sobolev argument, the bounds in (5.6) and (5.7) also hold in section 5.1 by choosing appropriate pseudodifferential cut-offs in (5.7).

Moreover, the same argument can be applied to Mass estimates on scales 5.2.

We get that
\[
\max (5.8) \quad \xi \geq 1 + \delta \quad \forall \xi \in \text{supp}(1 - \zeta) 
\]

Then, by the same integration by parts argument as in the proof of Proposition 5.1, one gets that
\[
\| (1 - \zeta(x', hD_{x'})) \varphi_h^{\nu} \|_{L^2} = O(h^\infty). \tag{5.6}
\]

Moreover, the same argument can be applied to \( \varphi_h^{\nu} \) resulting in the estimate
\[
\| (1 - \zeta(x', hD_{x'})) \varphi_h^{\nu} \|_{L^2} = O(h^\infty). \tag{5.7}
\]

By a Sobolev argument, the bounds in (5.6) and (5.7) also hold in \( C^k \)-norm for any \( k \in \mathbb{Z}^+ \). Since \( \epsilon_0 > 0 \) can be chosen arbitrarily small, this completes the proof. □

5.2. Mass estimates on scales \( O(h^\delta) \). We only need modify the argument in subsection 5.1 by choosing appropriate pseudodifferential cutoffs in \( Op_h^n(S^0_{\Sigma H, \delta})(T^*M) \) and then apply an analogous 2-microlocal parametrix construction. More precisely, for \( \delta \in [0, 1) \), we define
\[
\zeta(x', hD_{x'}) := (\chi_{\text{out}})^w_{h, \delta}(x', hD_{x'}). \tag{5.8}
\]

Then, by the same integration by parts argument as in the proof of Proposition 5.1, one gets that
\[
\| (\chi_{\text{out}})^w_{h, \delta} \varphi_h^{\nu} \|_{L^2} = O(h^\infty). \tag{5.9}
\]

6. Improved Neumann estimate

In this section, we revisit the computations leading to Theorem 2 and prove Theorem 1. We employ the same commutator argument, but in the first step, we do not use a small-scale frequency decomposition. That is, we let \( a^w \in Op_h(S^{0,0}) \) have principal symbol
\[
a(x, \xi) = \chi(x_n) \psi(\xi_n) \xi_n,
\]

where \( \chi \in C^\infty_0(\mathbb{R}) \) and \( \psi \in C^\infty(\mathbb{R}) \) are previously defined cutoff functions. Then, the same argument as in Theorem 2 (with \( \delta = 0 \)) gives:
\[
\frac{i}{h} \int_{M} [-h^2 \Delta - 1, a^w] \varphi_h \varphi_h^\infty dx
\]
\[
= \int_{H} (hD_n a^w \varphi_h)|_H \varphi_h^\infty|_H d\sigma_H
\]
\[
+ \int_{H} (a^w \varphi_h)|_H hD_n \varphi_h|_H d\sigma_H.
\]
Moreover, since the $\varphi_h$ are eigenfunctions, the left hand side is bounded. Using the fact that $\chi(0) - 1 = \chi'(0) = 0$ and $WF_h(\varphi_h) \subset \{|\xi|_g = 1\}$ (so that $\psi(\xi_n) = 1$ near the set $\{\xi_n: (x, \xi) \in WF_h(\varphi_h)\}$ combined with the fact that $\|a^{w}\|_{L^2 \to L^2} = O(1)$, one gets the estimate

$$
\int_H (1 + h^2 \Delta_H) \varphi_h^H \varphi_h^\perp d\sigma_H
+ \int_H |\varphi_h^{H,w}|^2 d\sigma_H = O(1).
$$

In order to bound the Neumann data from above, we therefore need to show the first term on the left hand side is semi-bounded below.

For this we now use our small scale decomposition. Let $\chi_{in}, \chi_{tan}, \chi_{out}$ be as before, and

$$
1 = (\chi_{in})_{h,\delta}^w + (\chi_{tan})_{h,\delta}^w + (\chi_{out})_{h,\delta}^w
$$

be the corresponding 2-microlocal partition of unity. We have

$$
\int_H (1 + h^2 \Delta_H) \varphi_h^H \varphi_h^\perp d\sigma_H
= \int_H (1 + h^2 \Delta_H) (\chi_{in})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
+ \int_H (1 + h^2 \Delta_H) (\chi_{tan})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
+ \int_H (1 + h^2 \Delta_H) (\chi_{out})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
= \int_H (1 + h^2 \Delta_H) (\chi_{in})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
+ \int_H (1 + h^2 \Delta_H) (\chi_{tan})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
+ \int_H (1 + h^2 \Delta_H) (\chi_{out})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H
+ O(h^\infty)
$$

from the computations leading to (5.8).

As estimated previously, on the support of $\chi_{in}$, we have $1 - R(x', 0, \xi') \geq h^\delta$, so that by the Gårding inequality,

$$
\int_H (1 + h^2 \Delta_H)(\chi_{in})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H \geq C_1 h^\delta \int_H (\chi_{in})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H.
$$

On the other hand, on the support of $\chi_{tan}$, we have $|1 - R(x', 0, \xi')| \leq C_2 h^\delta$, so that

$$
\left| \int_H (1 + h^2 \Delta_H)(\chi_{tan})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H \right| \leq C_2 h^\delta \left| \int_H (\chi_{tan})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H \right|.
$$

Combining these two estimates, we have

$$
\int_H (1 + h^2 \Delta_H) \varphi_h^H \varphi_h^\perp d\sigma_H
\geq C_1 h^\delta \int_H (\chi_{in})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H - C_2 h^\delta \left| \int_H (\chi_{tan})_{h,\delta}^w \varphi_h^H \varphi_h^\perp d\sigma_H \right| + O(h^\infty)
\geq -C h^\delta \int_H |\varphi_h^H|^2 d\sigma_H,
$$
since the exterior term is $O(h^\infty)$, so adding it back in is harmless. Employing again the $h^{-1/4}$ bound of Burq-Gérard-Tzvetkov [BGT], we get
\[ \int_H (1 + h^2 \Delta_H) \varphi_h^H \varphi_h^H d\sigma_H \geq -C h^{\delta-1/2}. \]
Choosing $\delta > 1/2$ gives
\[ -C h^{\delta-1/2} + \int_H |\varphi_h^H,\nu|^2 d\sigma_H \]
\[ \leq \int_H (1 + h^2 \Delta_H) \varphi_h^H \varphi_h^H d\sigma_H + \int_H |\varphi_h^H,\nu|^2 d\sigma_H \]
\[ = O(1), \]
or, rearranging,
\[ \int_H |\varphi_h^H,\nu|^2 d\sigma_H = O(1). \]

**Remark.** We observe that, since the $b_{2,in}$ term (3.10) is the only obstruction to proving the $O(1)$ bound, we obtain that, even though $b_{2,in} \in S^3_{\Sigma_H,\delta}$, its action on eigenfunctions is indeed bounded with
\[ \langle b_{2,in}^w(x, hD_x) \varphi_h, \varphi_h \rangle_{L^2(M^-)} = O(1). \]

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