CUTOFF RESOLVENT ESTIMATES AND THE SEMILINEAR
SCHRÖDINGER EQUATION

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Abstract. This paper shows how abstract resolvent estimates imply local
smoothing for solutions to the Schrödinger equation. If the resolvent estimate
has a loss when compared to the optimal, non-trapping estimate, there is
a corresponding loss in regularity in the local smoothing estimate. As an
application, we apply well-known techniques to obtain well-posedness results
for the semi-linear Schrödinger equation.

1. Introduction

In this short note we show how cutoff semiclassical resolvent estimates for the
Laplacian on a non-compact manifold, with spectral parameter on the real axis,
lead to well-posedness results for the semilinear Schrödinger equation. Motivated
by the requirements of [Chr3] and [BGT2], and the microlocal inverse estimates
of [Chr1, Chr2], we first prove a general theorem for a large class of resolvents.
Following the recent work of Nonnenmacher-Zworski [NoZw], we apply the general
theorem in the case there is a hyperbolic fractal trapped set.

Let \((M, g)\) be a Riemannian manifold of dimension \(n\) without boundary, with
(non-negative) Laplace-Beltrami operator \(-\Delta\) acting on functions. The Laplace-
Beltrami operator is an unbounded, essentially self-adjoint operator on \(L^2(M)\) with
domain \(H^2(M)\). We assume \((M, g)\) is asymptotically Euclidean in the sense of
[NoZw, (3.7)-(3.9)] and that the classical resolvent \((\Delta - (\lambda^2 + i\epsilon))^{-1}\)
obey a limiting absorption principle as \(\epsilon \to 0+, \lambda \neq 0\).

Our first result is that if we have cutoff semiclassical resolvent estimates with
a sufficiently small loss, then we have weighted smoothing for the Schrödinger
propagator with a loss. Let \(\rho_s\) be a smooth, non-vanishing weight function satisfying

\[
\rho_s(x) \equiv (d_g(x, x_0))^{-s},
\]

for some fixed \(x_0\) and \(x\) outside a compact set.

Theorem 1. Suppose for each compactly supported function \(\chi \in C_c^\infty(M)\) with
sufficiently small support, there is \(h_0 > 0\) such that the semi-classical Laplace-
Beltrami operator satisfies

\[
\| \chi (-\hbar^2 \Delta - E)^{-1} \chi u \|_{L^2(M)} \leq \frac{g(h)}{h}\| u \|_{L^2(M)}, \quad E > 0
\]

uniformly in \(0 < \hbar \leq h_0\), where \(g(h) \geq c_0 > 0, g(h) = o(h^{-1}). \) Then for each
\(T > 0\) and \(s > 1/2\), there is a constant \(C = C_{T,s} > 0\) such that

\[
\int_0^T \| \rho_s e^{i\hbar \Delta} u_0 \|_{H^{1/2-s}(M)}^2 dt \leq C \| u_0 \|_{L^2(M)}^2,
\]
where $\eta \geq 0$ satisfies
\begin{equation}
(1.4) \quad g(h)h^{2\eta} = O(1),
\end{equation}
and $\rho_s$ is given by (1.1).

The assumption that $(M, g)$ is asymptotically Euclidean is that there exists $R_0 > 0$ sufficiently large that, on each infinite branch of $M \setminus B(0, R_0)$, the semiclassical Laplacian $-h^2\Delta$ takes the form
\begin{equation}
-h^2\Delta|_{M \setminus B(0, R_0)} = \sum_{|\alpha| \leq 2} a_\alpha(x, h)(hD_x)^\alpha,
\end{equation}
with $a_\alpha(x, h)$ independent of $h$ for $|\alpha| = 2$,
\begin{equation}
\sum_{|\alpha| = 2} a_\alpha(x, h)(hD_x)^\alpha \geq C^{-1}|\xi|^2, \quad 0 < C < \infty, \quad \text{and}
\end{equation}
\begin{equation}
\sum_{|\alpha| \leq 2} a_\alpha(x, h)(hD_x)^\alpha \to |\xi|^2, \quad \text{as } |x| \to \infty \text{ uniformly in } h.
\end{equation}

In order to quote the results of [NoZw] we also need the following analyticity assumption:
\begin{equation}
\exists \theta_0 \in [0, \pi) \text{ such that the } a_\alpha(x, h) \text{ extend holomorphically to }
\end{equation}
\begin{equation}
\{ r\omega : \omega \in \mathbb{C}^n, \quad \text{dist}(\omega, S^n) < \epsilon, \quad r \in \mathbb{C}, \quad |r| \geq R_0, \quad \arg r \in [-\epsilon, \theta_0 + \epsilon) \}.
\end{equation}

As in [NoZw], the analyticity assumption immediately implies
\begin{equation}
\partial_x^\beta \left( \sum_{|\alpha| \leq 2} a_\alpha(x, h)\xi^\alpha - |\xi|^2 \right) = o(|x|^{-|\beta|}) \langle \xi \rangle^2, \quad |x| \to \infty.
\end{equation}

Recall the free Laplacian $(-\Delta_0 - \lambda^2)^{-1}$ on $\mathbb{R}^n$ has a holomorphic continuation from $\text{Im } \lambda > 0$ to $\lambda \in \mathbb{C}$ for $n \geq 3$ odd, and to the logarithmic covering space for $n$ even. This motivates the limiting absorption assumption, that
\begin{equation}
\lim_{\epsilon \to 0^+, \lambda \neq 0} \rho_s(-\Delta - (\lambda^2 + i\epsilon))^{-1}\rho_s
\end{equation}
exists as a bounded operator
\begin{equation}
L^2(M, d\text{vol}_g) \to L^2(M, d\text{vol}_g),
\end{equation}
provided $s > 1/2$. As in the free case, we allow a possible logarithmic singularity at $\lambda = 0$.

The problem of “local smoothing” estimates for the Schrödinger equation has a long history. The sharpest results to date are those of Doi [Doi] and Burq [Bur]. Doi proved if $M$ is asymptotically Euclidean, then one has the estimate
\begin{equation}
(1.5) \quad \int_0^T \|\chi e^{it\Delta}u_0\|_{H^{1/2}(M)}^2 \, dt \leq C\|u_0\|_{L^2(M)}^2
\end{equation}
for $\chi \in C_0^\infty(M)$ if and only if there are no trapped sets. Burq’s paper showed if there is trapping due to the presence of several convex obstacles in $\mathbb{R}^n$ satisfying certain assumptions, then one has the estimate (1.5) with the $H^{1/2}$ norm replaced by $H^{1/2-\eta}$ for $\eta > 0$. In [Chr3], the author considered an arbitrary, single trapped hyperbolic orbit. One of the goals of this paper is to use estimates obtained by Nonnenmacher-Zworski [NoZw] for fractal hyperbolic trapped sets to obtain similar results to [Chr3] for the semilinear Schrödinger equation. To that end we have the following corollary to Theorem 1.
Corollary 1.1. Assume \((M, g)\) admits a hyperbolic fractal trapped set, \(K_E\), in the energy level \(E > 0\) and that the topological pressure \(P_E(1/2) < 0\). Then \(-h^2\Delta - E\) satisfies (1.2) for some \(E > 0\) with \(g(h) = C\log(1/h)\), and for every \(\eta > 0\), \(T > 0\), and \(s > 1/2\), there exists a constant \(C = C_{P_E, \eta, T, s} > 0\) such that

\[
\int_0^T \|\rho_s e^{it\Delta} u_0\|_{L^{1/2 - \eta}(M)}^2 \, dt \leq C\|u_0\|^2_{L^2(M)}.
\]

We remark that the assumption \(P_E(1/2) < 0\) implies the trapped set \(K_E\) is filamentary or “thin” (see [NoZw] for definitions).

We consider the following semilinear Schrödinger equation problem:

\[
\begin{cases}
i\partial_t u + \Delta u = F(u) \text{ on } I \times M; \\u(0, x) = u_0(x),
\end{cases}
\]

where \(I \subset \mathbb{R}\) is an interval containing 0. Here the nonlinearity \(F\) satisfies

\[F(u) = G'(|u|^2)u,\]

and \(G : \mathbb{R} \to \mathbb{R}\) is at least \(C^3\) and satisfies

\[|G^{(k)}(r)| \leq C_k r^\beta - k,
\]

for some \(\beta \geq \frac{1}{2}\).

In §3 we prove a family of Strichartz-type estimates which will result in the following well-posedness theorem.

**Theorem 2.** Suppose \((M, g)\) satisfies the assumptions of the introduction, and set

\[
\delta = \frac{4\eta}{2\eta + 1} \geq 0.
\]

Then for each

\[
s > \frac{n}{2} - \frac{2}{\max \{2\beta - 2, 2\}} + \delta
\]

and each \(u_0 \in H^s(M)\) there exists \(p = \max \{2\beta - 2, 2\}\) and \(0 < T \leq 1\) such that (1.6) has a unique solution

\[
u \in C([-T, T]; H^s(M)) \cap L^p([-T, T]; L^\infty(M)).
\]

Moreover, the map \(u_0(x) \mapsto u(t, x) \in C([-T, T]; H^s(M))\) is Lipschitz continuous on bounded sets of \(H^s(M)\), and if \(\|u_0\|_{H^s}\) is bounded, \(T\) is bounded from below.

If, in addition, \((M, g)\) satisfies the assumptions of Corollary 1.1, \(n \leq 3\), \(\beta < 3\), and \(G(r) \to +\infty\) as \(r \to +\infty\), then \(u\) in (1.9) extends to a solution

\[
u \in C(( -\infty, \infty); H^1(M)) \cap L^p(( -\infty, \infty); L^\infty(M)).
\]

**Remark 1.2.** In particular, the cubic defocusing non-linear Schrödinger equation is globally \(H^1\)-well-posed in three dimensions with a fractal trapped hyperbolic set which is sufficiently filamentary. Of course other nonlinearities can be considered, but for simplicity we consider only these in this work.

**Acknowledgments.** This research was partially conducted during the period the author was employed by the Clay Mathematics Institute as a Liftoff Fellow.
2. Proof of Theorem 1

Since we are assuming \((-\Delta - z)^{-1}\) obeys a limiting absorption principle, we have
\[
\|\rho_s(-\Delta - (\tau + i\epsilon))^{-1}\rho_s\|_{L^2 \to L^2} \leq C_\epsilon
\]
for \(0 < \epsilon_0 \leq |\tau| \leq C\). For \(|\sigma| \geq C\) for some \(C > 0\), \(\sigma \in \mathbb{C}\) in a neighbourhood of the real axis, write
\[
-\Delta - \sigma = -\Delta - \frac{z}{h^2} = h^{-2}(-h^2\Delta - z),
\]
for
\[
\sigma = [E - \alpha, E + \alpha] + i[-c_0 h, c_0 h].
\]
Now
\[
(-h^2\Delta - z)
\]
is a Fredholm operator for \(z\) in the specified range, and hence the “gluing” techniques from [Vod] and [Chr3, §2] can be used to conclude for \(s > 1/2\),
\[
\rho_s(-h^2\Delta - z)^{-1},
\]
has a holomorphic extension to a slightly smaller neighbourhood in \(z\), and in particular,
\[
\|\rho_s(-h^2\Delta - E)^{-1}\rho_s\|_{L^2 \to L^2} \leq C \frac{g(h)}{h}.
\]
Rescaling, we have
\[
\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to L^2} \leq C \frac{g((\tau)^{1/2})}{(\tau)^{1/2}}, \quad \tau \in \mathcal{C}_{\pm \epsilon},
\]
where (see Figure 1)
\[
\mathcal{C}_{\pm \epsilon} = \{\tau \in \mathbb{R} : |\tau| \geq \epsilon\} \cup \{\tau \in \mathbb{C} : |\tau| = \epsilon, \pm \text{Im} \tau \geq 0\}.
\]
As in [Chr3] and [Bur], the following lemma follows from integration by parts and interpolation, together with the condition on \(\eta\), (1.4).

Lemma 2.1. With the notation and assumptions above, we have
\[
\|\rho_s(-\Delta - \tau)^{-1}\rho_s\|_{L^2 \to H^1} \leq C g((\tau)^{1/2}), \quad \tau \in \mathcal{C}_{\pm \epsilon},
\]
and for every $r \in [-1, 1]$,
\[ \|\rho_s(\Delta - \tau)^{-1}\rho_s\|_{H^r \to H^{1+r-n/2}} \leq C, \quad \tau \in C_{\pm \epsilon}. \]

Theorem 1 now follows from the standard “TT*” argument, letting $\epsilon \to 0$ in (2.1) (see [BGT2], the references cited therein, and [Chr3]).

The following Corollary uses interpolation with an $H^2$ estimate to replace the $H^{1/2-n}$ norm on the left hand side of (1.3) with $H^{1/2}$, and will be of use in §3. See [Chr3] for the details of the proof.

**Corollary 2.2.** Suppose $(M, g)$ satisfies the assumptions of Theorem 1. For each $T > 0$ and $s > 1/2$, there is a constant $C > 0$ such that
\[ \int_0^T \|\rho_s e^{it\Delta} u_0\|^2_{H^{1/2}(M)} dt \leq C \|u_0\|^2_{H^s(M)}, \]
where $\delta \geq 0$ is given by (1.7).

In particular, if $(M, g)$ satisfies the assumptions of Corollary 1.1, then for any $\delta > 0$, there is $C = C_\delta > 0$ such that (2.2) holds.

### 3. Strichartz-type Inequalities

In this section we give several families of Strichartz-type inequalities and prove Theorem 2. The statements and proofs are mostly adaptations of similar inequalities in [BGT2], so we leave out the proofs of these in the interest of space.

If we view $M \setminus U$, where $U$ is a neighbourhood of $K_E$, as a manifold with non-trapping geometry, we may apply the results of [HTW] or [BoTz] to a solution of the Schrödinger equation away from the trapping region, resulting in perfect Strichartz estimates. For this section we need (1.3) only with a compact cutoff $\chi$ instead of with the more general weight $\rho_s$.

**Proposition 3.1.** For every $0 < T \leq 1$ and each $\chi \in C_\infty(M)$ satisfying $\chi \equiv 1$ near $U$, there is a constant $C > 0$ such that
\[ \|(1 - \chi)u\|_{L^p([0, T])W^{s,q}(M)} \leq C \|u_0\|_{H^s(M)}, \]
where $u = e^{it\Delta} u_0$, $s \in [0, 1]$, and $(p, q)$, $p > 2$ satisfy
\[ \frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \]

**Remark 3.2.** In the sequel, wherever unambiguous, we will write
\[ L^p_t W^{s,q} := L^p([0, T])W^{s,q}(M) \]
and
\[ H^s := H^s(M). \]

**Proposition 3.3.** Suppose $(M, g)$ satisfies the assumptions of the Introduction, $u = e^{it\Delta} u_0$, and
\[ v = \int_0^t e^{i(t-\tau)\Delta} f(\tau) d\tau. \]
Then for each $0 < T \leq 1$ and $\delta \geq 0$ satisfying (1.7), we have the estimates
\[ \|u\|_{L^p_t W^{s-\delta,q}} \leq C \|u_0\|_{H^s}. \]
and
\[ \|v\|_{L^p_t W^{s-\frac{1}{q}}} \leq C\|f\|_{L^p_t H^s}, \]
where \( s \in [0, 1] \) and \((p, q)\), \( p > 2 \) satisfy the Euclidean scaling
\[ \frac{2}{p} + \frac{n}{q} = \frac{n}{2}. \]

The proof uses a local WKB expansion localized also in time to the scale of inverse frequency, followed by summing over frequency bands (see [Chr3] and [BGT1]). The only difference here is the explicit dependence of \( \delta \) on \( \eta \), which is related to the growth of the function \( g(h) \).

**Proof of Theorem 2.** The proof of Theorem 2 is a slight modification of the proof of Proposition 3.1 in [BGT1], but we include it here in the interest of completeness. Fix \( s \) satisfying 1.8 and choose \( p > \max\{2 \beta - 2, 2\} \) satisfying
\[ s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{1}{\max\{2 \beta - 2, 2\}} \]
where \( \delta \geq 0 \) satisfies (1.7). Set \( \sigma = s - \delta \) and
\[ Y_T = C([-T, T]; H^\sigma(M)) \cap L^p([-T, T]; W^{\sigma,q}(M)) \]
for
\[ \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \]
equipped with the norm
\[ \|u\|_{Y_T} = \max_{|t| \leq T} \|u(t)\|_{H^\sigma(M)} + \|u\|_{L^p_t W^{\sigma,q}}. \]

Let \( \Phi \) be the nonlinear functional
\[ \Phi(u) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-\tau)\Delta} F(u(\tau)) d\tau. \]

If we can show that \( \Phi : Y_T \to Y_T \) is a contraction on a ball in \( Y_T \) centered at 0 for sufficiently small \( T > 0 \), this will prove the first assertion of the Proposition, along with the Sobolev embedding
\[ W^{\sigma,q}(M) \subset L^\infty(M), \]
since \( \sigma > n/q \). From Proposition 3.3, we bound the \( W^\sigma \) part of the \( Y_T \) norm by the \( H^\sigma \) norm, giving
\[ \|\Phi(u)\|_{Y_T} \leq C \left( \|u_0\|_{H^\sigma} + \int_{-T}^T \|F(u(\tau))\|_{H^\sigma} d\tau \right) \]
\[ \leq C \left( \|u_0\|_{H^\sigma} + \int_{-T}^T (1 + |u(\tau)|)^{2\beta-2} \|u(\tau)\|_{H^\sigma} d\tau \right), \]
where the last inequality follows by our assumptions on the structure of \( F \). Applying Hölder’s inequality in time with \( \hat{p} = p/(2 \beta - 2) \) and \( \hat{q} \) satisfying
\[ \frac{1}{\hat{q}} + \frac{1}{\hat{p}} = 1 \]
gives
\[ \| \Phi(u) \|_{Y_T} \leq C \left( \| u_0 \|_{H^s} + T^\gamma \| u \|_{L_T^\infty H^s} \right) \]
where \( \gamma = 1/\tilde{q} > 0 \). Thus
\[ \| \Phi(u) \|_{Y_T} \leq C \left( \| u_0 \|_{H^s} + T^\gamma \left( \| u \|_{Y_T} + \| u \|_{Y_T}^{2/\tilde{q}} \right) \right). \]
Similarly, we have for \( u, v \in Y_T \),
\[
\begin{align*}
\| \Phi(u) - \Phi(v) \|_{Y_T} & \leq C T^\gamma \| u - v \|_{L_T^\infty H^s} \left( (1 + |u|) \| u \|_{L_T^\infty L^\infty}^{2/\tilde{q}} + (1 + |v|) \| v \|_{L_T^\infty L^\infty}^{2/\tilde{q}} \right), \\
& \leq C T^\gamma \| u - v \|_{Y_T} \left( (1 + |u|) \| u \|_{L_T^\infty L^\infty}^{2/\tilde{q}} + (1 + |v|) \| v \|_{L_T^\infty L^\infty}^{2/\tilde{q}} \right),
\end{align*}
\]
which is a contraction for sufficiently small \( T \). This concludes the proof of the first assertion in the Proposition.

To get the second assertion, we observe from (3.6) and the definition of \( Y_T \), if \( u \) and \( v \) are two solutions to (1.6) with initial data \( u_0 \) and \( u_1 \) respectively, so
\[ \tilde{\Phi}(v) = e^{it\Delta} u_1 - i \int_0^t e^{i(t-\tau)\Delta} F(v(\tau)) d\tau, \]
we have
\[ \max_{|t| \leq T} \| u(t) - v(t) \|_{H^s} = \max_{|t| \leq T} \| \Phi(u)(t) - \tilde{\Phi}(v)(t) \|_{H^s} \]
\[ \leq C \left( \| u_0 - u_1 \|_{H^s} + T^\gamma \max_{|t| \leq T} \| u(t) - v(t) \|_{H^s} \left( (1 + |u|) \| u \|_{L_T^\infty L^\infty}^{2/\tilde{q}} + (1 + |v|) \| v \|_{L_T^\infty L^\infty}^{2/\tilde{q}} \right) \right), \]
which, for \( T > 0 \) sufficiently small gives the Lipschitz continuity.

If \((M, g)\) satisfies the assumptions of Corollary 1.1, \( n \leq 3, \beta < 3, \) and \( G(r) \to +\infty \) as \( r \to +\infty \), we can take \( s \) and \( p \) satisfying \( p > \max\{2\beta - 2, 2\} \) and
\[ s > \frac{n}{2} - \frac{2}{p} + \delta \geq \frac{n}{2} - \frac{2}{\max\{2\beta - 2, 2\}} \]
for any \( \delta > 0 \). Then \( \sigma = s - \delta > q/n \) and the preceding argument holds. Finally, the proof of the global well-posedness now follows from the standard global well-posedness arguments from, for example, [Caz, Chapter 6].

References


[http://math.berkeley.edu/~zworski/nz3.ps.gz](http://math.berkeley.edu/~zworski/nz3.ps.gz)


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